# Robust mean field control: stochastic maximum principle and variational mean field games

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**Abstract**

We introduce a class of robust control problems formulated in min–max form, in which the principal agent is viewed as a central planner facing Nature. The agent's cost is a nonlinear function of all its possible realizations, encompassing in particular the mean field regime where the cost depends on the distribution of the states. In parallel, Nature favors the occurrence of outcomes that are least favorable to the agent, at an entropic cost. We establish existence and uniqueness of solutions under appropriate assumptions, including suitable convexity–concavity conditions, and derive a related stochastic maximum principle. We further address a corresponding class of robust variational mean field games in which the interaction term is subject to ambiguity, and prove existence and uniqueness of solutions.

Keywords: Robust mean field control, Stochastic maximum principle, Risk-averse control, Quadratic backward stochastic differential equation, Entropic penalties.

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## Contents



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# 1 Introduction

In this work we introduce a zero-sum non-local stochastic game in finite horizon between two players. Throughout, the first player is referred to as 'Nature' and the second one to as 'the central planner'.

**Formulation of the problem.** The problem is defined on a finite interval $[0, T]$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a $d$-dimensional Brownian motion $W = (W_t)_{t\in[0,T]}$ and an independent $n$-dimensional random variable $\eta$ representing the initial condition of the central planner. Here, $n \in \mathbb{N}^\star$ is the state dimension of the central planner and $d \in \mathbb{N}^\star$ the noise dimension to which the central planner is subjected. The $\mathbb{P}$-complete filtration generated by $(\eta, W)$ is denoted by $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$. On this probabilistic set-up, we consider the following inf-sup non-local (in the sense that the cost $\mathcal{G}$ below takes the entire random variables $q_T$ and $X_T^\psi$, and not only their realizations, as inputs) stochastic control problem:

$$\sup_{q\in\mathcal{Q}} \inf_{\psi\in\mathcal{A}} \mathcal{J}(q,\psi), \quad \mathcal{J}(q,\psi) := \mathcal{R}(q,\psi) - \mathcal{S}(q), \tag{P}$$

where

$$\mathcal{R}(q,\psi) := \mathcal{G}\big(q_T, X_T^\psi\big) + \mathbb{E}\left[\int_0^T q_s \ell(s,\psi_s)\mathrm{d}s\right], \tag{1}$$

$$\mathcal{S}(q) := \mathbb{E}\left[\int_0^T q_s f^\star(s, Y_s^\star, Z_s^\star)\mathrm{d}s\right]. \tag{2}$$

In this formulation, equilibria are sought over open loop controls. Nature optimizes with respect to $q \in \mathcal{Q}$ and the central planner with respect to $\psi \in \mathcal{A}$, where the admissible sets $\mathcal{Q}$ and $\mathcal{A}$ can be roughly described as follows:

- The set $\mathcal{Q}$ is a class of $\mathbb{F}$-progressively measurable, positive-valued processes with finite entropy $\mathcal{S}$, accounting for changes in the historical measure $\mathbb{P}$ under uncertainty from Nature (here and throughout, 'positive' is understood in the sense of strictly positive). Precisely, a process $q$ belongs to $\mathcal{Q}$ if

$$\mathcal{S}(q) < +\infty, \tag{3}$$

$$\text{and} \quad q_t = 1 + \int_0^t q_s Y_s^\star \mathrm{d}s + \int_0^t q_s Z_s^\star \cdot \mathrm{d}W_s, \quad t \in [0,T], \tag{4}$$

  where $Y^\star = (Y_t^\star)_{t\in[0,T]}$ and $Z^\star = (Z_t^\star)_{t\in[0,T]}$ are two $\mathbb{F}$-progressively measurable processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. The process $q$ admits an explicit expression in terms of $Y^\star$ and $Z^\star$:

$$q_t = e^{\int_0^t Y_s^\star \mathrm{d}s} \mathcal{E}_t\left(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s\right), \tag{5}$$

  where $(\mathcal{E}_t(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s))_{t\in[0,T]}$ is the stochastic exponential associated to $Z^\star$. From now on, we denote $q_T\mathbb{P}$ the equivalent (non-normalized) measure defined as $\int_A q_T \mathrm{d}\mathbb{P}$ for all $A \in \mathcal{F}$. When $|Y^\star| = 0$, $q = (q_t)_{t\in[0,T]}$ is a Doléans-Dade exponential and defines an equivalent probability measure $q_T\mathbb{P}$. When $|Y^\star| > 0$, $q$ defines a collection of equivalent non-normalized measures $(q_t\mathbb{P})_{t\in[0,T]}$, which we refer to as 'discounted measures'. We refer the reader to Appendix A for more details about the representation of $q$.

- The set $\mathcal{A}$ consists in a class of $\mathbb{F}$-progressively-measurable, $\mathbb{R}^n$-valued processes $\psi = (\psi_t)_{t\in[0,T]}$ such that

$$\mathcal{S}^\star(\psi) < +\infty, \quad \mathcal{S}^\star(\psi) := \sup_{q\in\mathcal{Q}} \left\{ \mathbb{E}\left[\int_0^T q_s |\psi_s|^2 \mathrm{d}s\right] - \gamma \mathcal{S}(q) \right\}. \tag{6}$$

  The coefficient $\gamma$ has to be fixed carefully and will be clearly defined in the Assumption A5 below, but we already mention that it should depend on the other data of the problem. For a given control $\psi \in \mathcal{A}$, the state $X^\psi = (X_t^\psi)_{t\in[0,T]}$ of the central planner is the solution to

$$\mathrm{d}X_t = b(t, X_t, \psi_t)\mathrm{d}t + \sigma(t, \psi_t)\mathrm{d}W_t, \quad X_0 = \eta, \tag{7}$$

  where the drift $b\colon \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and the volatility $\sigma\colon \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are possibly random. Implicitly, $b$ and $\sigma$ are required to be $\mathbb{F}$-progressively measurable. The precise assumptions on the two of them will be clarified later in the article; see Subsection 3.1. In particular the state equation (7) will be assumed to be linear, but we keep it under general form for the exposition.

Returning to (1) and (2), $\ell\colon \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is referred to as the running cost. The coefficient $f^\star\colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is called the 'convex dual' driver (for reasons explained below). This function $f^\star$ is typically viewed as (a perturbation of) the square of its last argument. The function $\mathcal{G}$ represents the terminal cost. In its most general form, it is defined as a (measurable) real-valued mapping on $\Omega \times L^1(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}_+) \times L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$. This formulation encompasses mean field functions with arguments such as $\mathbb{Q} \circ (X_T)^{-1}$ with $\mathbb{Q} = \mathcal{E}_T(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s)\mathbb{P}$; this example motivates the term central planner for the player optimizing over $\psi$. The problem is thus called non-local, since the functional $\mathcal{G}$ requires the full information on the terminal random variables $(q_T, X_T^\psi)$ to be evaluated. In principle, we could incorporate a running cost of a similar structure in (1), but for the sake of simplicity and clarity, we will omit this term from the remainder of the article.

The cost functions can be interpreted as follows: when the central planner chooses a strategy $\psi$, Nature tries to adjust the historical probability $\mathbb{P}$ by weighting it with $q$ in the worst possible way for the planner, thus maximizing the cost $\mathcal{R}(q, \psi)$. Conversely, once the weighting $q$ is chosen, the planner aims to select the best strategy $\psi$ to minimize $\mathcal{R}(q, \psi)$. This is an 'almost classic' stochastic control problem, depending on the form of the terminal cost $\mathcal{G}$. When $\mathcal{G}(q_T, X_T^\psi)$ is written as an expectation $\mathbb{E}[q_T g(X_T)]$, the planner solves a standard problem under the discounted measure $q\mathbb{P}$. When $\mathcal{G}(q_T, X_T^\psi)$ takes the form $G((q_T\mathbb{P}) \circ (X_T^\psi)^{-1})$, with $G$ being a cost function defined on the space $\mathcal{M}_+(\mathbb{R}^n)$ of positive measures on $\mathbb{R}^n$, the planner solves a mean field control problem under the measures $q\mathbb{P}$. In both cases, the running cost $\ell$ can be chosen to be quadratic or to grow quadratically in $\psi$.

**A preview: risk averse control problem and BSDEs.** To better understand the problem (P), we focus in this paragraph on the first of the two cases above, namely, we assume that there exists a function $g\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathcal{G}\big(q_T, X_T^\psi\big) \coloneqq \mathbb{E}\left[q_T g(X_T^\psi)\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{\int_0^T Y_s^\star \mathrm{d}s} g(X_T^\psi)\right],$$

where $\mathbb{Q}$ is the equivalent probability measure defined by $\mathbb{Q} = \mathcal{E}_T(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s)\mathbb{P}$. Here the term $Y^\star$ can be understood as an actualization rate, which might be negative. In this framework, the problem (P) becomes

$$\sup_{q \in \mathcal{Q}} \inf_{\psi \in \mathcal{A}} J(q, \psi), \quad J(q, \psi) \coloneqq R(q, \psi) - \mathcal{S}(q), \tag{$\mathrm{P_L}$}$$

where

$$R(q, \psi) \coloneqq \mathbb{E}^{\mathbb{Q}}\left[e^{\int_0^T Y_s^\star \mathrm{d}s} g(X_T^\psi) + \int_0^T e^{\int_0^s Y_u^\star \mathrm{d}u} \ell(s, \psi_s) \mathrm{d}s\right].$$

When $\psi \in \mathcal{A}$ is fixed, the penalty $\mathcal{S}(q)$ prevents Nature from choosing a singular measure (relative to the historical probability $\mathbb{P}$) that would only assign weight to the worst outcome for the central planner. In fact, the problem solved by Nature coincides with the risk-aversion problem presented in [86, Chapter 6.4], with the key difference being that the variable $Z^\star$ is bounded in [86], which greatly simplifies the analysis. In particular, [86] provides a representation of the value of the problem (corresponding here to the problem solved by Nature) in the form of a Backward Stochastic Differential Equation (BSDE) driven by coefficients with at most linear growth. In our framework, this BSDE may become quadratic, as explained in the next paragraph.

To further fix the ideas about the 'linear' problem ($\mathrm{P_L}$), assume that $f^\star(t, y^\star, z^\star) = \frac{1}{2}|z^\star|^2$, for all $(t, y^\star, z^\star) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, and $Y^\star \equiv 0$. Because the actualization rate $Y^\star$ is null, $q_T$ is the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$, that is $\mathrm{d}\mathbb{Q} = q_T \mathrm{d}\mathbb{P}$. Due to the specific form of $f^\star$, the penalty $\mathcal{S}(q)$ is equal to $\mathrm{H}(\mathbb{Q}|\mathbb{P})$ where $\mathrm{H}(\mathbb{Q}|\mathbb{P}) := \mathbb{E}^{\mathbb{Q}}[\ln(\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})]$ denotes the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$. Then, the cost simplifies to

$$J(q,\psi) = \mathbb{E}^{\mathbb{Q}}\left[g(X_T^\psi) + \int_0^T \ell(s,\psi_s)\mathrm{d}s\right] - \mathrm{H}(\mathbb{Q}|\mathbb{P}). \tag{8}$$

Nature's problem then coincides with a maximization problem that frequently appears in large deviation theory. Indeed, the Donsker–Varadhan variational formula provides an interpretation of Nature's optimal value as the log-Laplace transform of a cost function defined on the Wiener space, as seen in works like [29, 55]. This problem has a long history in economic and finance literature [39, 63], and can be found under different names (ambiguity, robust or risk sensitive control problem, depending on the interpretation) We also refer to the recent contribution [20] for a systematic analysis of entropy-penalized stochastic optimal control problems.

When considering an optimizer $q \in \mathcal{Q}$ for Nature's problem in ($\mathrm{P_L}$), the remaining central planner minimization problem over $\psi$ can be reformulated as a control problem over BSDEs:

$$\inf_{\psi \in \mathcal{A}} \mathbb{E}\left[Y_0^\psi\right],$$

where $(Y,Z)$ is the solution to,

$$-\mathrm{d}Y_t = \left(f(t,Y_t,Z_t) + \ell(t,\psi_t)\right)\mathrm{d}t - Z_t \cdot \mathrm{d}W_t, \quad Y_T = g(X_T^\psi), \tag{9}$$

and $f$ is the Fenchel transform of $f^\star$ (see (2)). This connection is presented in [86, Chapter 6.4] in the particular case of linear growth drivers $f$. When $f$ is quadratic in the variable $z$, as considered throughout the remainder of the article, solving the BSDE in equation (9) becomes more challenging. The study of quadratic BSDEs began with the seminal work of [70] on equations driven by bounded terminal conditions. For a comprehensive presentation of the standard theory, see [99], which includes additional references. Subsequent research has extended the results on existence and uniqueness to unbounded terminal conditions, under the assumption of finite exponential order moments [24, 25, 50]. We will return to these references in the core of the article, as our analysis is typically conducted in the context where the terminal value $g$ is unbounded.

The BSDE in equation (9) can be interpreted as a nonlinear conditional expectation, specifically a $g$-expectation [85]. When the criterion $J$ is given by (8), that is, when $f(s,y,z) = \frac{1}{2}|z|^2$, the first component $Y$ of the BSDE is known in the literature as the entropic risk measure of the cost $g(X_T^\psi) + \int_0^T \ell(\psi_t)\mathrm{d}t$. Entropic risk measures have been extensively studied in the $L^\infty$ case, i.e., for bounded costs, see [7].

**First contribution: From risk neutral to robust mean field control.** The main objective of our paper is twofold: first, from a technical perspective, to relax the growth conditions of the various cost functionals in the problem ($\mathrm{P_L}$); and second, from a modeling perspective, to consider a mean field version, whose general form is given in (P). In this regard, the problem (P) encompasses not only mean field control problems with risk aversion but, more generally, problems in which the central planner is subject to uncertainty, here perceived as an adverse action of Nature. A series of examples are provided in Subsections 3.2 and 4.2 to illustrate these concepts.

In the risk-neutral case, stochastic mean field control problems are typically introduced as the limiting behavior of optimal control problems defined over large interacting particle systems. In these settings, a central planner seeks to optimize an objective function that depends on the collective dynamics of the particles. This class of problems has attracted significant attention in recent years [4, 18, 19, 27, 45, 46, 53, 73, 76]. For a comprehensive introduction to the subject, we refer to [11, 35, 36].

In this article, we establish the stochastic maximum principle for the problem (P). The stochastic maximum principle is a powerful tool for solving stochastic control problems, first introduced by [72] and further developed by [14], [65], [83], and [97]. It plays a central role in the stochastic mean field control and mean fied game literature [28, 35, 36]. The standard theory of the stochastic maximum principle applies to risk-neutral control problems and is typically formulated in an $L^2$ framework, where both the state variables and the adjoint processes are assumed to belong to $L^2$. To establish the stochastic maximum principle, three key steps are typically followed: first, proving the existence of a solution to the control problem [64]; second, deriving the necessary conditions [84]; and third, demonstrating the sufficient conditions, which can be shown using a simple verification argument.

Here, we move beyond the scope of the standard theory for two main reasons, which align with the two primary objectives of our work. The first is to address a mean field problem with a risk-averse min-max structure. Extensions of the stochastic maximum principle to risk-averse problems have been studied in the context of optimal control of Forward-Backward Stochastic Differential Equations (FBSDEs). For example, see [82] for cases with linear growth drivers and jumps. The second objective is to allow the terminal condition $g$ to be unbounded. While bounded terminal conditions enable the use of the BMO theory for quadratic BSDEs [66], such assumptions are too restrictive for some applications. Moreover, they are rather incompatible with the convexity constraints typically required in the sufficient condition of the maximum principle. One natural approach to obtain stronger exponential integrability properties on $\psi$, compatible with those required in the theory of quadratic BSDEs, would be to follow the methodology of [41, 42] and work within an Orlicz space framework. Indeed, Orlicz spaces generalize $L^p$ spaces and, in particular, include random variables with finite exponential moments of arbitrary order, together with their dual space, which consists of random variables with finite entropy H. Such a dual space would be a natural candidate for carrying the variable $q$. That said, adopting this approach in our setting would require working with a quadratic driver of the form $f(t, y, z) = \frac{1}{\gamma}|z|^2$. For $\gamma$ large enough, this would provide the level of exponential integrability needed to apply the theory of quadratic BSDEs. However, for small values of $\gamma$, to the best of our knowledge, the stochastic maximum principle is not available even in this simpler setting. In contrast, our analysis goes one step further: the driver $f$ is only assumed to have at most quadratic growth, and may in fact exhibit subquadratic growth. Our strategy is to extend the duality inherent to Orlicz spaces of random variables to a setting involving dual spaces of stochastic processes. This perspective motivates the introduction of the mappings $\mathcal{S}$ and $\mathcal{S}^\star$, which define the admissible sets $\mathcal{Q}$ and $\mathcal{A}$.

The first major contribution of this article is the proof of the stochastic maximum principle for the problem (P). Under appropriate concavity-convexity conditions, we show that this problem has a unique solution, where the minimizer is fully characterized by the solution of a FBSDE. To establish this result, we begin by considering a constrained version of (P), for which we identify a topological structure ensuring semi-continuity, convexity/concavity, and compactness of the criterion $\mathcal{J}$ in each variable. This preliminary analysis allows us to apply Sion's min–max theorem and to deduce the existence of a saddle point for the constrained problem. We then relax the constraints by showing that there exists a level at which they are in fact non-binding, which in turn yields the existence of a saddle point for the original problem (P). The necessary and sufficient optimality conditions are obtained by coupling the first-order conditions associated with Nature's problem and the central planner's problem. It is worth emphasizing that the necessary conditions provided by the stochastic maximum principle require solving FBSDEs that go beyond the scope of the standard theory. The sufficient conditions ensure the uniqueness of these solutions. In the course of the analysis, we revisit the connection between entropy-type optimization problems and quadratic BSDEs with unbounded terminal conditions, a connection previously established for linear functionals $\mathcal{G}$ in [50].

**Second contribution: Robust mean field control and variational mean field games.** Mean field control (MFC) problems constitute a class of stochastic optimal control problems in which both the system dynamics and the associated cost functional.may depend on the distribution of the controlled state process. Such problems naturally arise in the modeling of large populations of weakly interacting particles, where the influence of each individual is mediated through the empirical distribution of the population. Typical applications can be found in economics, statistical physics, and mathematical finance. In recent years, these problems have attracted significant attention; see, for instance, [28, 30, 34, 53, 73, 87], among many others. When treated from a probabilistic perspective, they are often addressed via the stochastic maximum principle.

In this article, we introduce a *robust* version of this problem, where the measure encoding the mean field interaction is biased by Nature. For a real-valued function $G$, defined on the space of non-negative measures on $\mathbb{R}^n$, we thus consider the min-max problem

$$\inf_{\psi\in\mathcal{A}}\sup_{q\in\mathcal{Q}}\left\{G(\mathbb{Q}^q_X)+\mathbb{E}\left[\int_0^T q_s\ell(s,\psi_s)\mathrm{d}s\right]-\mathcal{S}(q)\right\}, \tag{MFC}$$

where $(X^\psi_t)_{t\in[0,T]}$ is the solution to the controlled stochastic differential equation (7) and

$$\mathbb{Q}^q_X := \mathbb{Q}\circ X^{-1},\quad \mathbb{Q}=\exp\left(\int_0^T Y^\star_s\mathrm{d}s\right)\mathcal{E}_T\left(\int_0^\cdot Z^\star_s\cdot\mathrm{d}W_s\right)\mathbb{P}.$$

This problem is a specification of the problem (P) when $\mathcal{G}(q,X)=G(\mathbb{Q}^q_X)$. Building upon the stochastic maximum principle established for (P), we derive the stochastic maximum principle for the problem (MFC) under the assumption that the mapping $G$ is Lions differentiable, Lions convex and flat concave.

In addition, we also study a variational mean field game (MFG) problem. In contrast to MFC problems, which are cooperative in essence, MFGs are competitive problems. They are defined over a continuum of players whose interactions arise through a mean field functional. The theory of MFGs was introduced independently in [67] and [74, 75], and has since been extensively developed; see, for instance, [12, 16, 31, 32, 35, 36, 37]. MFGs have found numerous applications in economics and finance [1, 33, 57, 80], environmental studies [69, 77], and electricity markets [3], to name just a few. We also refer to [35, 36] for a comprehensive monograph. The classical theory of MFGs typically considers risk-neutral agents. A natural extension is therefore to investigate models with risk-averse agents. Several approaches have been proposed in this direction, each relying on different ways of incorporating risk aversion into the representative agent's cost functional. Risk-sensitive MFGs [79, 95] introduce criteria depending on the variance of the state, while risk-averse MFGs [40, 56, 59] incorporate risk measures directly into the cost functional. MFGs in which agents optimize a worst-case criterion, using $H^\infty$ control techniques, were introduced in [8]. The theory of MFGs is closely related to that of MFC, particularly through variational (or potential) MFGs, which form a special class of MFGs. In brief, a variational MFG can be formulated as the first-order optimality conditions of a stochastic MFC problem [9, 10, 17, 26, 61, 62]. In particular, any solution to the MFC problem yields an equilibrium of the associated variational game. Moreover, when the MFC problem is strictly convex and coercive, the corresponding variational MFG admits a unique solution.

In this article, we study the following MFG problem. Given a non-negative measure $\mu$ on $\mathbb{R}^n$, representing the mean field coupling, a representative agent (in the continuum) minimizes a risk-averse objective functional

$$\inf_{\psi\in\mathcal{A}}\sup_{q\in\mathcal{Q}} \mathcal{J}[\mu](q,\psi) := \mathbb{E}\left[q_T\frac{\delta G}{\delta\mu}(X_T^\psi,\mu_T) + \int_0^T q_s\ell(s,\psi_s)\mathrm{d}s\right] - \mathcal{S}(q), \tag{P$_\mu$}$$

where the controlled state process $(X_t^\psi)_{t\in[0,T]}$ satisfies the dynamics given in (7), and $\delta G/\delta\mu$ is the so-called flat derivative of $G$, see Section 4 for a reminder. For a saddle point $(\psi,q)\in\mathcal{A}\times\mathcal{Q}$, the mean field equilibrium condition is defined as follows: the measure $\mu$ is required to coincide with the law of the terminal state $X_T^\psi$ under the probability measure $\mathbb{Q}^q$ induced by Nature, that is,

$$\mu = \mathbb{Q}^q_{X^\psi} := \mathbb{Q}^q\circ(X_T^\psi)^{-1}, \quad \mathbb{Q}^q = \mathcal{E}\left(\int_0^\cdot Z_s^\star\cdot\mathrm{d}W_s\right). \tag{MFG-eq}$$

In other words, the MFG problem consists in finding a triple $(q,\psi,\mu)$, with $(q,\psi)\in\mathcal{Q}\times\mathcal{A}$ and $\mu$ being a non-negative measure, such that

$$\mathcal{J}[\mu](q,\psi) = \inf_{\psi'\in\mathcal{A}}\sup_{q'\in\mathcal{Q}}\mathcal{J}[\mu](q',\psi'), \quad \mu = \mathbb{Q}^q\circ(X_T^\psi)^{-1}. \tag{MFG}$$

The optimization problem faced by the representative player can be interpreted as a risk-averse (non mean field) control problem. To find the optimal strategy, the representative agent solves a risk-averse stochastic control problem that falls within the scope of (non-mean field) control problems addressed in this work.

Under the same regularity and concavity–convexity assumptions on $G$ as those used in the analysis of the robust MFC (MFC), we establish the existence and uniqueness of an equilibrium, which ultimately coincides with the solution of (MFC). As such, this article is the first to identify a variational structure for risk-averse MFGs. The analysis of such robust MFGs is pursued further in our companion work [49], where we go beyond the variational setting.

**Organization of the article.** The article is organized as follows. In Section 2, we introduce the main notations and definitions used throughout the paper. Section

3 contains our main result, Theorem 10, which establishes a stochastic maximum principle for the problem (P), together with first examples of applications. Section 4 is devoted to the mean field setting. There, we establish the existence and uniqueness of solutions to a class of robust mean field control problems in Corollary 15, and we then consider a related class of robust variational mean field games, proving existence and uniqueness of equilibria in Corollary 16. Additional examples are provided in Subsection 4.2. Finally, Section 5 is dedicated to the proof of Theorem 10.

## 2 Notations

In this section, we introduce the main notations used in the article. Throughout, we work on the same filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ as in the definition of the problem (P).

**Spaces of random variables and random processes.** We begin by introducing the spaces of variables and stochastic processes on which our analysis relies. Unless otherwise stated, all notations are understood to be with respect to the probability measure $\mathbb{P}$. When a different measure, say $\mathbb{Q}$, is used, this will be made explicit. For example, in the context of the first example below, we will write $L^p(\dots, \mathbb{Q})$ to indicate the underlying measure. Moreover, for each of the spaces defined below, we will often omit the notation $\mathbb{R}^k$ when $k = 1$.

*Usual random variable spaces.* For a given $k \in \mathbb{N}^*$ and for each $t \in [0, T]$, we denote by $L^0(\mathcal{F}_t, \mathbb{R}^k)$ the set of $\mathbb{R}^k$ valued and $\mathcal{F}_t$-measurable random variables (r.v.'s in short). And then, we define the sets

- $L^p(\mathcal{F}_t, \mathbb{R}^k)$ of r.v.'s $X \in L^0(\mathcal{F}_t, \mathbb{R}^d)$ s.t. $\|X\|_{L^p(\mathcal{F}_t,\mathbb{R}^k)} \coloneqq \mathbb{E}[|X|^p] < +\infty$, for $p < +\infty$,
- $L^\infty(\mathcal{F}_t, \mathbb{R}^k)$ of r.v.'s $X \in L^0(\mathcal{F}_t, \mathbb{R}^k)$ s.t. $\|X\|_{L^\infty(\mathcal{F}_t,\mathbb{R}^k)} \coloneqq \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{i\in\{1,\dots,d\}} |X^i(\omega)| < +\infty$.

*Usual random process spaces.* We denote by $L^0(\mathbb{F}, \mathbb{R}^k)$ the space of $\mathbb{F}$-progressively measurable random processes (r.p.'s in short) with values in $\mathbb{R}^k$, and by $S^0(\mathbb{F}, \mathbb{R}^k)$ the subset of $L^0(\mathbb{F}, \mathbb{R}^k)$ comprising processes with continuous trajectories. We define the sets

- $L^p(\mathbb{F}, \mathbb{R}^k)$ of r.p.'s $X \in L^0(\mathbb{F}, \mathbb{R}^k)$ s.t. $\|X\|_{L^p(\mathbb{F},\mathbb{R}^k)} \coloneqq \mathbb{E}\left[\left(\int_0^T |X_t|^p \mathrm{d}t\right)^{1/p}\right] < +\infty$, for $p < +\infty$,
- $M^p(\mathbb{F}, \mathbb{R}^k)$ of r.p.'s $X \in L^0(\mathbb{F}, \mathbb{R}^k)$ s.t. $\|X\|_{M^p(\mathbb{F},\mathbb{R}^k)} \coloneqq \mathbb{E}\left[\left(\int_0^T |X_t|^2 \mathrm{d}t\right)^{p/2}\right] < +\infty$,
- $L^\infty(\mathbb{F}, \mathbb{R}^d)$ of r.p.'s $X \in L^0(\mathbb{F}, \mathbb{R}^k)$ s.t. $\|X\|_{L^\infty(\mathbb{F},\mathbb{R}^k)} \coloneqq \sup_{t\in[0,T]} \|X_t\|_{L^\infty(\mathcal{F}_t,\mathbb{R}^k)} < +\infty$,
- $S^p(\mathbb{F}, \mathbb{R}^k)$ of r.p.'s $X \in S^0(\mathbb{F}, \mathbb{R}^k)$ s.t. $\|X\|_{S^p(\mathbb{F},\mathbb{R}^k)} \coloneqq \mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^p\right] < +\infty$.

- $D(\mathbb{F},\mathbb{R}^k)$ of r.p.'s $X \in S^0(\mathbb{F},\mathbb{R}^k)$ such that the family $(|X_\tau|)_\tau$, with $\tau$ running over the set of $[0,T]$-valued $\mathbb{F}$-stopping times, is uniformly integrable.

The class $D(\mathbb{F},\mathbb{R}^k)$, which is the least standard among the above classes, was introduced in [52, Definition 20].

Moreover, for a process $X = (X_t)_{t\in[0,T]}$, with continuous trajectories and with values in $\mathbb{R}^k$, we denote by $(X_t^* := \sup_{s\in[0,t]} |X_s|)_{t\in[0,T]}$ the running maximum of the norm of $X$.

*Orlicz* spaces. Following (3), we define the entropy function $h\colon \mathbb{R}_+ \to \mathbb{R}$:

$$h(x) := x(\ln(x) - 1), \tag{10}$$

together with the two sets

- $L\log L(\mathcal{F}_T)$ of r.v.'s $X \in L^0(\mathcal{F}_T)$ s.t. $\mathbb{E}\left[h(X_T)\right] < +\infty$,
- $L\log L(\mathbb{F})$ of r.p.'s $X \in L^0(\mathcal{F}_t)$ s.t. $\sup_{t\in[0,T]} \mathbb{E}\left[h(X_t)\right] < +\infty$.

For any $X \in L^0(\mathcal{F}_T)$, with non-negative values, we call entropic risk measure of level $\vartheta > 0$ of $X$ the quantity

$$\rho_\vartheta[X] := \frac{1}{\vartheta} \ln \mathbb{E}\left[\exp(\vartheta X)\right], \tag{11}$$

which makes it possible to define the sets

- $L^{p,\vartheta}_{\exp}(\mathcal{F}_t,\mathbb{R}^k)$ of r.v.'s $X \in L^0(\mathcal{F}_t,\mathbb{R}^d)$ s.t. $\rho_\vartheta[|X|^p] < +\infty$,
- $L^{p,\vartheta}_{\exp}(\mathbb{F},\mathbb{R}^k)$ of r.p.'s $X \in L^0(\mathbb{F},\mathbb{R}^k)$ s.t. $\rho_\vartheta\left[\int_0^T |X_s|^p \mathrm{d}s\right] < +\infty$,
- $S^{p,\vartheta}_{\exp}(\mathbb{F},\mathbb{R}^k)$ of r.p.'s $X \in L^0(\mathbb{F},\mathbb{R}^k)$ s.t. $\rho_\vartheta\left[|X_T^*|^p\right] < +\infty$,

for $k \in \mathbb{N}^*$, $p > 0$ and $t \in [0,T]$. When $X \in L^0(\mathcal{F}_T)$ and $|X| \in L^{1,\vartheta}_{\exp}(\mathcal{F}_T)$, the right-hand side (11) still makes sense and we can define $\rho_\vartheta[X]$ accordingly. Moreover, for $p > 0$, we denote $L^p_{\exp}(\mathcal{F}_t,\mathbb{R}^k)$ the set of random variables $X \in L^0(\mathcal{F}_t,\mathbb{R}^k)$ such that $X \in L^{p,\vartheta}_{\exp}(\mathcal{F}_t,\mathbb{R}^k)$ for some $\vartheta > 0$. The sets $L^p_{\exp}(\mathbb{F},\mathbb{R}^d)$ and $S^p_{\exp}(\mathbb{F},\mathbb{R}^d)$ are defined in an analogous way.

**Spaces of measures.** For a metric space $(\mathcal{X}, d)$, we call $\mathcal{B}(\mathcal{X})$ its Borel $\sigma$-field, $\mathcal{P}(\mathcal{X})$ the set of probability measures on $\mathcal{X}$, and $\mathcal{M}(\mathcal{X})$ the set of finite non-negative measures on $\mathcal{X}$. Let $p \geq 1$ we define the sets

- $\mathcal{P}_p(\mathcal{X})$ of $\mu \in \mathcal{P}(\mathcal{X})$ s.t. $\int_\mathcal{X} d(x_0,x)^p \mathrm{d}\mu(x) < +\infty$ for some $x_0 \in \mathcal{X}$,
- $\mathcal{M}_p(\mathcal{X})$ of $\mu \in \mathcal{M}(\mathcal{X})$ s.t. $\int_\mathcal{X} d(x_0,x)^p \mathrm{d}\mu(x) < +\infty$ for some $x_0 \in \mathcal{X}$.

For any finite measure $\mathbb{Q}$ on $\Omega$ and measurable mapping $X\colon \Omega \to \mathcal{X}$, we denote $\mathbb{Q}\circ X^{-1}$, or $\mathbb{Q}_X$, the image measure of $\mathbb{Q}$ by $X$. When $f$ is a non-normalized non-negative measurable function on $\Omega$, we denote by $f\mathbb{P}$ the equivalent non-normalized measure $\mathbb{Q} : \mathcal{F} \ni A \mapsto \mathbb{Q}(A) := \int_A f \mathrm{d}\mathbb{P}$. In particular, for $X$ as before, $(f\mathbb{P})_X$ stands for the image of $f\mathbb{P}$ by $X$.

Lastly, for $\mu^1, \mu^2 \in \mathcal{P}(\mathcal{X})$, we define the relative entropy by

$$\mathrm{H}(\mu^1|\mu^2) := \int_{\mathcal{X}} \ln\left(\frac{\mathrm{d}\mu^1}{\mathrm{d}\mu^2}\right) \mathrm{d}\mu^1,$$

if $\mu^1$ is absolutely continuous with respect to $\mu^2$, and we set $\mathrm{H}(\mu^1|\mu^2) = +\infty$ otherwise. Additional material on the metric structures of $\mathcal{P}(\mathcal{X})$ and $\mathcal{M}(\mathcal{X})$ is introduced in Subsection 4.1.

**Duality.** We end this section with duality results.

*Fenchel transform.* The following duality is used repeatedly all along the article. By Fenchel duality, we have, for any $x^\star \in \mathbb{R}$ and $x \in \mathbb{R}_+$ (recalling the definition of $h$ in (10)),

$$\exp(x^\star) + h(x) \geq x^\star x. \tag{12}$$

We often make use of the duality inequality (12), when reformulated in the form

$$\begin{aligned} x^\star x = (\vartheta x^\star)\, \frac{x}{\vartheta} &\leq h\left(\frac{x}{\vartheta}\right) + \exp(\vartheta x^\star) \\ &= \frac{1}{\vartheta} h(x) - \ln(\vartheta) x + \exp(\vartheta x^\star), \end{aligned} \tag{13}$$

for all $\vartheta > 0$ and any $x, x^\star > 0$.

*Duality between $\mathcal{S}$ and $\mathcal{S}^\star$.* By definition of $\mathcal{S}$ and $\mathcal{S}^\star$ in (2) and (6), we have for any $\mathbb{F}$-progressively measurable processes $q$ and $\zeta$, valued in $\mathbb{R}$,

$$\mathcal{S}(q) + \mathcal{S}^\star(\zeta) \geq \frac{1}{\gamma} \mathbb{E}\left[\int_0^T q_s |\zeta_s|^2 \mathrm{d}s\right], \tag{14}$$

where $\mathcal{S}(q)$ and $\mathcal{S}^\star(\zeta)$ might take infinite values. This inequality is a direct consequence of the definition of $\mathcal{S}^\star$. Equality holds whenever

$$q \in \arg\max_{q' \in \mathcal{Q}} \left\{ \mathbb{E}\left[\int_0^T q_s |\zeta_s|^2 \mathrm{d}s\right] - \gamma \mathcal{S}(q) \right\}.$$

*Dual Donsker-Varadhan variational formula.* Let $\mu \in \mathcal{P}(\mathbb{R}^n)$ and $\vartheta > 0$ be such that $\int_{\mathbb{R}^n} \exp(\alpha \cdot x) \mathrm{d}\mu(x) < +\infty$ for all $|\alpha| \leq \vartheta$. If there is a finite constant $L > 0$ such that $|k(x)| \leq L(1 + |x|)$ for any $x \in \mathbb{R}^n$ then

$$-\ln \int_{\mathbb{R}^n} \exp\left(-k(x)\right) \mathrm{d}\mu(x) = \inf_{m \in \mathcal{P}(\mathbb{R}^n) : H(m|\mu) < \infty} \left\{ \mathrm{H}(m|\mu) + \int_{\mathbb{R}^n} k(x) \mathrm{d}m(x) \right\}. \tag{15}$$

This formula can be found in [29, Proposition 2.3]. It remains valid even when $\mu$ does not satisfy exponential integrability, provided that $k$ is bounded from above, with no assumption on its growth from below.

**Miscellaneous.** Throughout the article, we use a generic constant $C > 0$ that depends only on the data of the problem. The value of $C$ may change from line to line. As for the data of the problem themselves, they are introduced and specified in the assumptions section.

When $x$ and $y$ are vectors of finite dimension, $x \cdot y$ denotes the scalar product between $x$ and $y$.

# 3 Stochastic maximum principle

In this section, we establish the stochastic maximum principle for the (non–mean field) problem (P). The section is organized into two main subsections. The main result, Theorem 10, which presents the stochastic maximum principle for the problem (P), is stated in Subsection 3.1. Its proof relies on an application of Sion's min–max theorem (recalled below), together with the stochastic maximum principles for both Nature's problem and the central planner's problem. These two problems are treated independently in Section 5. Two application examples are discussed in Subsection 3.2.

**Theorem 1** (Sion [93])**.** *Let $M$ be a compact convex subset of a linear topological space and $N$ a convex subset of a linear topological space. Let $v\colon M \times N \to \mathbb{R}$ be such that*

1. *$v(\cdot, y)$ is lower semi-continuous and convex on $M$ for each $y \in N$,*
2. *$v(x, \cdot)$ is upper semi-continuous and concave on $N$ for each $x \in M$.*

*Then we have*

$$\min_{x \in M} \sup_{y \in N} v(x, y) = \sup_{y \in N} \min_{x \in M} v(x, y)$$

*and supremum is attained whenever $N$ is compact.*

## 3.1 Main result

We first present the assumptions used throughout the paper, even though some intermediate results are stated under weaker conditions. Additional assumptions are introduced in Section 4 when discussing the mean field setting.

**Assumptions.** The assumptions are stated in terms of two constants, $L > 0$ and $r \in \{0, 1\}$. They also make use of the notion of progressively-measurable field: for a metric space $(\mathcal{X}, d)$ and an integer $k \in \mathbb{N}^*$, a random field $\mathcal{G} : \Omega \times [0, T] \times \mathcal{X} \to \mathbb{R}^k$ is said to be progressively-measurable if, for any $t \in [0, T]$, its restriction to $\Omega \times [0, t] \times \mathcal{X}$ is $\mathcal{F}_t \otimes \mathcal{B}([0, t]) \otimes \mathcal{B}(\mathcal{X})/\mathcal{B}(\mathbb{R}^k)$ measurable.

A1 *Initial condition and drift.* The initial condition $\eta$ in (7) belongs to $L^\infty(\mathcal{F}_0, \mathbb{R}^n)$, i.e.

$$\|\eta\|_{L^\infty(\mathcal{F}_0, \mathbb{R}^n)} < +\infty.$$

The drift $b\colon \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is linear and of separated form

$$b(t, x, \psi) = a_t + b_t x + c_t \psi,$$

where $a$, $b$ and $c$ belong respectively to $L^\infty(\mathbb{F}, \mathbb{R}^n)$, $L^\infty(\mathbb{F}, \mathbb{R}^{n\times n})$ and $L^\infty(\mathbb{F}, \mathbb{R}^{n\times n})$, i.e. $\|a\|_{L^\infty(\mathbb{F}, \mathbb{R}^n)} + \|b\|_{L^\infty(\mathbb{F}, \mathbb{R}^{n\times n})} + \|c\|_{L^\infty(\mathbb{F}, \mathbb{R}^{n\times n})} < +\infty$. In particular, $b$ is a progressively-measurable random field.

A2 *Volatility.* The volatility $\sigma\colon \Omega \times [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ is linear in the control variable. Precisely, the $n \times d$ entries of the matrix $\sigma$ are of the form

$$(\sigma(t, \psi))_{i,j} = (\nu_t)_{i,j} + r(\sigma_t)_{i,j,k}\psi_k,$$

$(i,j,k) \in \{1,\dots,n\} \times \{1,\dots,d\} \times \{1,\dots,n\}$, with $\nu$ and $\sigma$ belonging respectively to $L^\infty(\mathbb{F},\mathbb{R}^{n\times d})$ and $L^\infty(\mathbb{F},\mathbb{R}^{n\times d})$ and satisfying $\|\nu\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})} + \|\sigma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d\times n})} \le L$.

A3 *Driver.* The driver $f\colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is progressively-measurable, convex and twice differentiable with respect to its last two variables, the corresponding derivatives of order 2 are bounded by $L$. Moreover, there exist two constants $\alpha, \beta \ge 0$ such that $t$,

$$f(t,y,z) \le |f^0_t| + \alpha|y| + \frac{\beta}{2}|z|^2, \quad (y,z) \in \mathbb{R}\times\mathbb{R}^d,$$

where $f^0 \coloneqq f(0,0) \in L^\infty(\mathbb{F})$.

A4 *Running cost.* The running cost $\ell\colon \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is progressively-measurable and twice differentiable in the last variable. It satisfies

$$\big(\nabla_\psi \ell(t,\psi) - \nabla_\psi \ell(t,\psi')\big)\cdot(\psi-\psi') \ge \frac{1}{L}|\psi-\psi'|^2, \quad |D^2_\psi \ell(t,\psi)| \le L,$$

and $|\ell(t,0)| \le L$ for any $t \in [0,T]$ and $\psi,\psi' \in \mathbb{R}^n$. In particular, $\ell$ is strongly convex in the last variable, with a quadratic growth, uniformly in the other variables.

A5 *Coefficients.* The coefficient $\gamma$ in (6) is chosen as

$$\begin{aligned}\gamma =& 8\beta \max(1,L) e^{\alpha T} \|\Gamma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})} \|\Gamma^{-1}\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})} \\ &\times \Big(\|\nu\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})} + 12\max(1,L)e^{\alpha T}\|\sigma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d\times n})}\Big),\end{aligned}$$

where $\Gamma$ is the resolvent of the linear ODE driven by $b$, i.e. the solution to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_t = b_t\Gamma_t, \quad t\in[0,T], \quad \Gamma_0 = I_n,$$

with $I_n$ standing for the $n\times n$ identity matrix.

Moreover, when $r=0$, the following smallness condition is satisfied:

$$4\beta e^{\alpha T} L \|\Gamma\|^2_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})} \|\Gamma^{-1}\|^2_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})} \|\nu\|^2_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})} T < 1,$$

The choice of $\gamma$ is discussed in Remark 5 below, and the smallness condition in Remark 7.

A6 *Growth of the mapping $\mathcal{G}$ and its derivatives.* Denoting by $\mathscr{G}$ the set of pairs $(q,X)$ of $\mathcal{F}_T$-measurable random variables with values in $\mathbb{R}_+ \times \mathbb{R}^n$ such that $\mathbb{E}[q|X|^{2-r}] < +\infty$, the cost $\mathcal{G}$ is a real-valued function on $\mathscr{G}$. Together with some mappings $\delta_q\mathcal{G} : \mathscr{G} \to L^0(\mathcal{F},\mathbb{R})$ and $\delta_X\mathcal{G} : \mathscr{G} \to L^0(\mathcal{F},\mathbb{R}^d)$, which are interpreted below as derivatives of $\mathcal{G}$ in the directions $q$ and $X$ respectively, it satisfies the growth properties:

$$\begin{aligned}|\mathcal{G}(q,X)| &\le L\left(1+\mathbb{E}\left[(1+q)|X|^{2-r}\right]\right),\\ -L\left(1+|X|+\mathbb{E}\left[q|X|^{2-r}\right]\right) \le \delta_q\mathcal{G}(q,X) &\le L\left(1+|X|^{2-r}+\mathbb{E}\left[q|X|^{2-r}\right]\right),\\ |\delta_X\mathcal{G}(q,X)| &\le Lq\left(1+|X|^{1-r}+\mathbb{E}\left[q|X|^{2-r}\right]\right).\end{aligned}$$

A7 *First order Taylor expansion of the mapping $\mathcal{G}$.* With $\mathscr{G}$, $\delta_q\mathcal{G}$ and $\delta_X\mathcal{G}$ as in the previous condition, the mapping $\mathcal{G}$ admits the following two first order expansions in $X$ and $q$ respectively:

$$\begin{aligned}\mathcal{G}(q,X') &= \mathcal{G}(q,X) + \mathbb{E}\left[\delta_X\mathcal{G}(q,X)\cdot(X'-X)\right] + O\left(\mathbb{E}\left[q|X'-X|^2\right]\right)\\ \mathcal{G}(q',X) &= \mathcal{G}(q,X) + \mathbb{E}\left[\delta_q\mathcal{G}(q,X)(q'-q)\right] + o\left(\mathbb{E}\left[(1+|X|^{2-r})|q'-q|\right]\right),\end{aligned}$$

where $|O(r)| \le cr$ for a constant $c$ that only depends on $(q, X', X)$ via (any bound for) $\mathbb{E}[q|X|^{2-r}]$ and $\mathbb{E}[q|X'|^{2-r}]$, and where $|o(r)| \le \eta(r)r$ for a function $\eta$ that tends 0 with $r$ and that only depends on $(q, q', X)$ via (any bound for) $\mathbb{E}[q|X|^{2-r}]$ and $\mathbb{E}[q|X'|^{2-r}]$.

A8 *Concavity-convexity of the mapping $\mathcal{G}$.* The mapping $\mathcal{G}$ is concave with respect to the variable $q$ and convex with respect to the variable $X$, i.e., for any $(q, X)$, $(q^1, X^1)$ and $(q^2, X^2)$ in $\mathscr{G}$, with $\mathscr{G}$ as in (A6), and for any $\theta \in [0, 1]$.

$$\begin{aligned}\mathcal{G}\left(\theta q^1 + (1-\theta)q^2, X\right) &\ge \theta\mathcal{G}(q^1,X) + (1-\theta)\mathcal{G}(q^2,X),\\ \mathcal{G}\left(q, \theta X^1 + (1-\theta)X^2\right) &\le \theta\mathcal{G}(q,X^1) + (1-\theta)\mathcal{G}(q,X^2).\end{aligned}$$

**Comments and examples.** We provide several comments and examples to clarify the assumptions.

*Remark* 2. *Lower bound on the dual driver.* In (2), $f^\star\colon \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ is the Fenchel transform of the driver $f$ with respect to its variables $(y, z)$, i.e.,

$$f^\star(t,y^\star,z^\star) \coloneqq \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d} \left\{\langle (y^\star,z^\star),(y,z)\rangle - f(t,y,z)\right\}.$$

Since $f$ is continuous in $(y, z)$, the supremum in the definition of $f^\star$ can be reduced to a supremum over a countable set. We easily deduce that $f^\star$ is progressively-measurable.

Moreover, because $f$ is twice differentiable in $(y, z)$ with bounded second-order derivatives, see A3, $f^\star$ is $c$-strongly convex with respect to its last two variables, for a constant $c > 0$, see for instance [49] for an explicit proof.

Assumption A3 also implies that, for any $(t, y^\star, z^\star) \in [0, T]\times\mathbb{R}\times\mathbb{R}^d$,

$$f^\star(t,y^\star,z^\star) \ge -|f_t^0| + \chi_{\mathcal{B}}(y^\star/\alpha) + \frac{1}{2\beta}|z^\star|^2, \tag{16}$$

where $\chi_{\mathcal{B}}$ denotes the indicator function of the unit ball $\mathcal{B} \coloneqq \{x \in \mathbb{R},\ |x| \le 1\}$, i.e.,

$$\chi_{\mathcal{B}}(y^\star) = \begin{cases} 0, & \text{if } y^\star \in \mathcal{B},\\ +\infty, & \text{otherwise.}\end{cases}$$

Indeed by the growth condition in A3, we have, for any $t \in [0, T]$, $(y^\star, z^\star) \in \mathbb{R}\times\mathbb{R}^d$ and $(y, z) \in \mathbb{R}\times\mathbb{R}^d$,

$$\begin{aligned}f^\star(t,y^\star,z^\star) &\ge \langle (y^\star,z^\star),(y,z)\rangle - f(t,y,z)\\ &\ge \langle (y^\star,z^\star),(y,z)\rangle - |f_t^0| - \alpha|y| - \frac{\beta}{2}|z|^2.\end{aligned}$$

Taking, on both sides, the supremum with respect to $(y, z) \in \mathbb{R}\times\mathbb{R}^d$ and recalling that the absolute value $|\cdot|$ and the indicator function $\chi_{\mathcal{B}}$ are in duality, we get (16).

*Example* 3. *Mean field structure of $\mathcal{G}$.* As we already mentioned, the problem addressed in this section is not of mean field type. It is only in Section 4 that we clarify our application to the mean field case, by considering cost functions $\mathcal{G}$ of the form

$$\mathcal{G}(q, X) = G\left((q\mathbb{P})_X\right),$$

where $(q\mathbb{P})_X$ denotes the law of $X$ under $q\mathbb{P}$, assuming that $q$ is a non-negative random variable, and $G$ is a cost function defined on the space of non-negative measures.

*Remark* 4. *Linearity of the state equation.* The linearity of the state equation, as guaranteed by Assumptions A1 and A2, ensures the concavity of the mapping $\mathcal{Q} \ni q \mapsto \mathcal{J}(q, \psi)$ and the convexity of the mapping $\mathcal{A} \ni \psi \mapsto \mathcal{J}(q, \psi)$, which are proved in Proposition 20 and Lemma 22, respectively. Additionally, the assumption that the volatility is independent of the state variable is crucial for guaranteeing the existence of a finite exponential moment of $L|X_T^{0,*}|$, where $X^0$ denotes the solution to the state equation when $\psi \equiv 0$. Such a property would generally fail if the volatility depended linearly on the state variable. See also Remark 7 for further comments on the exponential integrability of $X_T^{0,*}$.

*Remark* 5. *On the constant $\gamma$.* The choice of $\gamma$, as specified in A5, stems from Lemma 32. Roughly speaking, the latter provides an a priori bound on the component $\psi$ of any saddle point $(q, \psi)$ of (P). This bound is formulated in terms of a bound on $\mathcal{S}^\star(\psi)$ and therefore requires an appropriate choice of $\gamma$.

*Remark* 6. *On the constant $r$.* The volatility is controlled when $r = 1$ and uncontrolled when $r = 0$. As suggested in the previous remark, the state variable $X$ has finite exponential moments of sufficiently small order when the volatility is controlled ($r = 1$). When the volatility is uncontrolled ($r = 0$), stronger results can be established, showing that $X$ has finite quadratic exponential moments of small order, as detailed in Lemma 42. The fact that the integrability properties are stronger when $r = 0$ explains why the growth assumption A6 is more general in this case.

*Remark* 7. *On the smallness condition on $\|\nu\|^2_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})}$.* Part of our analysis relies on an a priori bound for the component $q$ of an arbitrary saddle point $(q, \psi)$ of (P). This bound is established in Lemma 26. To make the proof work, we require the existence of some $\psi \in \mathcal{A}$ (in fact, for simplicity, we choose $\psi \equiv 0$) such that $L|X_T^{\psi,*}|^{2-r}$ admits an exponential moment of sufficiently large order $\upsilon$ (with $\upsilon$ depending explicitly on the other parameters in the assumptions). When $r = 1$, the random variable $|X_T^{0,*}|^{2-r} = |X_T^{0,*}|$ admits exponential moments of all orders. When $r = 0$, the random variable $L|X_T^{0,*}|^{2-r} = L|X_T^{0,*}|^2$ admits an exponential moment of order $\upsilon$ provided the smallness condition stated in A5 is in force (see Lemma 42).

It must be stressed that we require stronger integrability properties of $|X_T^{0,*}|$ in the case $r = 0$ than in the case $r = 1$, due to the growth conditions imposed on $\mathcal{G}$. If we were to work with the same growth conditions as in the case $r = 1$, the smallness condition would no longer be needed.

Finally, we note that the smallness condition imposed here is reminiscent of the integrability assumptions appearing in the analysis of quadratic BSDEs with unbounded terminal data; see, for instance, [25, 50]. This is not surprising, since the characterization of the saddle points of (P) resulting from our analysis (see Theorem 10) relies on a forward–backward SDE that may be quadratic (if $f$ is). In this respect, it is worth emphasizing that our smallness condition is not imposed at the level of the saddle point itself, but rather at the level of a single controlled trajectory. As such, it is more explicit and easier to verify.

*Remark* 8. *On the running cost $\ell$.* In our analysis, the running cost $\ell$ is assumed to be independent of the state variable. This assumption may be restrictive for certain applications. However, our approach also allows one to consider a running cost $\ell' \colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that depends on the state variable $x$ and is of separated form. More precisely, there exists a function $c' \colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\ell'(t,x,\psi) = c'(t,x) + \ell(t,\psi).$$

One has to assume that $c$ is convex with respect to its second variable, and satisfies the growth condition

$$|c'(t,x)| \leq L(1+|x|^{2-r}).$$

The latter implies that

$$\mathbb{E}\left[\int_0^T q_s \ell'(s, X_s^0, 0)\mathrm{d}s\right] \leq L\left(1 + \mathbb{E}\left[\int_0^T (1+q_s)|X_s^0|^{2-r}\mathrm{d}s\right]\right),$$

which is, in particular, enough to reproduce the proofs of the two key Lemmas 26 and 32 (up to an adaptation of the two conditions in A5).

Inequality (16) has an important consequence, which we formalize in the following statement:

**Lemma 9.** *Let $q = (q_t)_{t\in[0,T]}$ be an $\mathbb{F}$-progressively measurable positive-valued continuous process such that $\mathcal{S}(q) < +\infty$. Then,*

$$\mathbb{P} \otimes \mathrm{Leb}_{[0,T]}\left(\{(\omega,t) \in \Omega \times [0,T],\ |Y_t^\star| > \alpha\}\right) = 0. \tag{17}$$

*Moreover,*

$$\mathbb{P}\left(\left\{\int_0^T |Z_s^\star|^2 \mathrm{d}s < +\infty\right\}\right) = 1, \tag{18}$$

*and $(\mathcal{E}_t(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s))_{t\in[0,T]}$ in (5) is a 'true' martingale.*

While the first claim, (17), is quite obvious, the second one, (18), is more subtle. Indeed, we deduce from (16) that

$$\mathbb{P}\left(\left\{\int_0^T q_s |Z_s^\star|^2 \mathrm{d}s < +\infty\right\}\right) = 1.$$

And then, it is by continuity and strict positivity of $q$ that (18) follows. In particular, it must be observed that the stochastic integral in (5) is necessarily well-defined. It is then clear that (4) and (5) are equivalent: starting from (4), one obtains (5) by applying Itô's formula, while the converse implication also follows from Itô's formula, applied to the process $(\ln(q_t))_{t\in[0,T]}$, which is well defined since $q$ takes strictly positive values. The fact that $(\mathcal{E}_t(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s))_{t\in[0,T]}$ is a true martingale is a follows from Lemma 39, proved in Appendix A.

**Main result.** The main result of this section is presented in Theorem 10 below. We introduce the pre-Hamiltonian of the system $\mathcal{H}\colon \Omega \times [0,T] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$,

$$\begin{aligned}\mathcal{H}(t,q,y^\star,z^\star,y,z,x,\psi,p,k) := q(yy^\star + z\cdot z^\star - f^\star(t,y^\star,z^\star) + \ell(t,\psi)) \\ +p{\cdot}b(t,x,\psi) + \mathrm{Tr}(k\sigma^\top(t,\psi)).\end{aligned} \tag{19}$$

Although the pre-Hamiltonian $\mathcal{H}$ explicitly appears in Isaac's condition discussed in Remark 11 below, for the purpose of our analysis, it is more convenient to split it into two parts, each corresponding to the pre-Hamiltonian used by either the central planner or Nature:

$$\begin{aligned}F(t,q,y^\star,z^\star,y,z,\psi) &:= q\left(yy^\star + z\cdot z^\star - f^\star(t,y^\star,z^\star) + \ell(t,\psi)\right), \\ H(t,x,\psi,p,k,q) &:= q\ell(t,\psi) + p\cdot b(t,x,\psi) + \mathrm{Tr}(k\sigma^\top(t,\psi)).\end{aligned} \tag{20}$$

Given $q \in \mathcal{Q}$, we say that a tuple $(\psi,p,k,X)$ satisfies the first order condition ($\mathrm{Opt_C}$) for the central planner problem if $(\psi,p,k,X)$ is a solution to

$$\begin{cases} -\mathrm{d}p_t &= \nabla_x H(t,X_t,\psi_t,p_t,k_t,q_t)\mathrm{d}t - k_t\mathrm{d}W_t, \quad p_T = \delta_X\mathcal{G}(q_T,X_T^\psi), \\ \mathrm{d}X_t &= b(t,X_t,\psi_t)\mathrm{d}t + \sigma(t,\psi_t)\mathrm{d}W_t, \quad X_0 = \eta, \\ \psi_t &\in \arg\min_\alpha H(t,X_t,\alpha,p_t,k_t,q_t), \quad \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-a.s.} \end{cases} \tag{$\mathrm{Opt_C}$}$$

The first equation is interpreted as the adjoint equation for the central planner, the second equation as the state equation, and the last equation as the optimality condition. Because the last equation couples the two preceding equations, the system above is an FBSDE. The wordings 'first order condition' and 'optimality condition' are fully justified by the statement of Theorem 10 below.

Given $\psi \in \mathcal{A}$, we say that a tuple $(Y,Z,q)$ satisfies the first order condition ($\mathrm{Opt_N}$) for Nature problem if $(Y,Z,q)$ is a solution to

$$\begin{cases} -\mathrm{d}Y_t &= \partial_q F(t,q_t,Y_t^\star,Z_t^\star,Y_t,Z_t,\psi_t)\mathrm{d}t - Z_t\cdot\mathrm{d}W_t, \quad Y_T = \delta_q\mathcal{G}(q_T,X_T^\psi), \\ \mathrm{d}q_t &= q_tY_t^\star\mathrm{d}t + q_tZ_t^\star\cdot\mathrm{d}W_t, \quad q_0 = 1, \\ (Y_t^\star,Z_t^\star) &\in \arg\max_{(Y^{\star\prime},Z^{\star\prime})} F(t,q_t,Y^{\star\prime},Z^{\star\prime},Y_t,Z_t,\psi_t), \quad \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-a.s.} \end{cases} \tag{$\mathrm{Opt_N}$}$$

The first equation is interpreted as the adjoint equation for Nature, the second equation describes the dynamics of the control variable, and the last equation is the optimality condition. Similar to the previous one, this system of equations is also an FBSDE.

The two systems ($\mathrm{Opt_C}$) and ($\mathrm{Opt_N}$) above are presented in an abstract form. To clarify the result, we now give an explicit formulation using the concrete expressions of the coefficients. We start with the system ($\mathrm{Opt_C}$). Computing the gradient of the Hamiltonian $\nabla_x H$ and the optimality condition, we have

$$\begin{cases} -\mathrm{d}p_t &= b_t^\top p_t\mathrm{d}t - k_t\mathrm{d}W_t, \quad p_T = \delta_X\mathcal{G}(q_T,X_T^\psi), \\ \mathrm{d}X_t &= b(t,X_t,\psi_t)\mathrm{d}t + \sigma(t,\psi_t)\mathrm{d}W_t, \quad X_0 = \eta, \\ 0 &= q_t\nabla_\psi\ell(t,\psi_t) + c_t^\top p_t + r\mathrm{Tr}(\sigma_t^\top k_t), \quad \mathrm{d}\mathbb{P}\otimes\mathrm{d}t\text{-a.s,} \end{cases} \tag{21}$$

where we denote, by convention,

$$\mathrm{Tr}\left(\sigma_t^\top k_t\right) = \left(\sum_{i=1}^{n}\sum_{j=1}^{d}(\sigma_t)_{i,j,\ell}(k_t)_{i,j}\right)_{\ell=1,\dots,d}. \tag{22}$$

We now turn to the system ($\mathrm{Opt_N}$). The optimality condition is given by

$$(Y_t^\star, Z_t^\star) = (\partial_y f(t, Y_t, Z_t), \partial_z f(t, Y_t, Z_t)). \tag{23}$$

Computing the derivative of the Hamiltonian $\partial_q F$ and plugging the optimality condition into the backward equation, the latter equation becomes a (possibly quadratic) BSDE by Fenchel's duality

$$\begin{cases} -\mathrm{d}Y_t &= (f(t, Y_t, Z_t) + \ell(t, \psi_t))\mathrm{d}t - Z_t \cdot \mathrm{d}W_t, \quad Y_T = \delta_q \mathcal{G}(q_T, X_T^\psi), \\ \mathrm{d}q_t &= q_t Y_t^\star \mathrm{d}t + q_t Z_t^\star \cdot \mathrm{d}W_t, \quad q_0 = 1. \end{cases} \tag{24}$$

For the purpose of analyzing these two systems, we define the following two spaces. The first is the space of solutions to the system ($\mathrm{Opt_C}$) given $\psi$ within a certain sub-level set of $\mathcal{A}$ (which will be specified when necessary), and the second is the space of solutions to ($\mathrm{Opt_N}$) given $q \in \mathcal{Q}$:

$$\mathscr{A} := \mathcal{A} \times D(\mathbb{F}) \times (\bigcap_{\beta\in(0,1)} M^\beta(\mathbb{F}, \mathbb{R}^d)) \times S^{2-r}(\mathbb{F}, \mathbb{R}^n, \mathbb{Q}), \tag{25}$$

$$\mathscr{Q} := \left\{(q, Y, Z) \in \mathcal{Q} \times D(\mathbb{F}, \mathbb{Q}) \times (\bigcap_{\beta\in(0,1)} M^\beta(\mathbb{F}, \mathbb{R}^d, \mathbb{Q}))\right\}, \tag{26}$$

where $\mathbb{Q}$ in the first line is the measure $q_T\mathbb{P}$. Here is now our main statement regarding the inf-sup mean field stochastic control problem (P).

**Theorem 10.** *There exists a unique saddle point $(\bar{q}, \bar{\psi}) \in \mathcal{Q} \times \mathcal{A}$ to the problem (P), i.e.*

$$\min_{\psi\in\mathcal{A}} \max_{q\in\mathcal{Q}} \mathcal{J}(q, \psi) = \max_{q\in\mathcal{Q}} \min_{\psi\in\mathcal{A}} \mathcal{J}(q, \psi) = \mathcal{J}(\bar{q}, \bar{\psi}).$$

*Moreover, if a pair $(\psi, q) \in \mathcal{A} \times \mathcal{Q}$ is a solution to the problem (P), then the tuples $(\psi, p, k, X)$, obtained by solving in $\mathscr{A}$ the two decoupled equations in ($\mathrm{Opt_N}$), and $(q, Y, Z)$, obtained by solving in $\mathscr{Q}$ the two decoupled equations in ($\mathrm{Opt_C}$), satisfy the optimality conditions in ($\mathrm{Opt_N}$) and ($\mathrm{Opt_C}$) respectively. Conversely, if $(\psi, p, k, X, q, Y, Z) \in \mathscr{A} \times \mathscr{Q}$ is a solution to ($\mathrm{Opt_C}$)-($\mathrm{Opt_N}$), then the pair $(\psi, q) \in \mathcal{A} \times \mathcal{Q}$ is the solution to the problem (P).*

We provide a sketch of the proof based on the results established in the core of the article. We believe this presentation will help the reader gain a global overview of the structure of the arguments.

The strategy relies on introducing two truncation parameters. For $c_1, c_2 > 0$, we define the two following sets:

$$\mathcal{Q}_{c_1} := \{q \in \mathcal{Q},\ \mathcal{S}(q) \le c_1\}, \tag{27}$$

$$\mathcal{A}_{c_2} := \{\psi \in L^2(\mathbb{F}, \mathbb{R}^n),\ \mathcal{S}^\star(\psi) \le c_2\}. \tag{28}$$

Accordingly, we define the following min-max problem, analogous to the problem (P), but with the above two sets as restricted admissible sets:

$$\sup_{q\in\mathcal{Q}_{c_1}} \inf_{\psi\in\mathcal{A}_{c_2}} \mathcal{J}(q, \psi). \tag{P'}$$

*Proof. Step 1: Existence of a saddle point to* (P'). The problem (P') is studied in Subsection 5.1. Existence of a saddle point is established in Lemma 17.

*Step 2: Interior solutions.* By definition, any saddle point $(q, \psi)$ to (P') satisfies

$$\mathcal{J}(q, 0) \geq \mathcal{J}(q, \psi) \geq \mathcal{J}(q^0, \psi), \tag{29}$$

where $q^0 = (q_t^0 = 1)_{t\in[0,T]} \in \mathcal{Q}_{c_1}$ denotes the solution to $q_t = 1 + \int_0^t q_s Y_s^\star \mathrm{d}s + \int_0^t q_s Z_s^\star \cdot \mathrm{d}W_s$ with $(Y^\star, Z^\star) \equiv (0, 0)$. Then, by the two forthcoming Lemmas 26 and 32, there exist two constants $c_1', c_2' > 0$ only depending on the data and independent of $c_1$ and $c_2$ such that (more precisely $c_2'$ depends on $c_1'$, which is only depending on the data) such that

$$(q, \psi) \in \left(\mathcal{Q}_{c_1} \cap \mathcal{Q}_{c_1'}\right) \times \left(\mathcal{A}_{c_2} \cap \mathcal{A}_{c_2'}\right).$$

Now choosing $c_1$ and $c_2$ such that $c_1 > c_1'$ and $c_2 > c_2'$ yields that $(q, \psi) \in \mathcal{Q}_{c_1'} \times \mathcal{A}_{c_2'}$ and thus $(q, \psi)$ is an interior solution to the problem (P'), in the sense that $\mathcal{S}(q)$ and $\mathcal{S}^\star(\psi)$ are respectively strictly less than $c_1$ and $c_2$.

*Step 3: Nature's problem.* Let $(q, \psi) \in \mathcal{Q}_{c_1} \times \mathcal{A}_{c_2}$ be a saddle point to (P'), for $c_1 > c_1'$. By the previous step, $q$ lies in the interior of $\mathcal{Q}_{c_1}$. Theorem 25 (whose statement and proof are the main objectives of Subsection 5.2.1 below) says that $q$ is a maximizer of the problem

$$\sup_{q' \in \mathcal{Q}_{c_1}} \mathcal{J}(q', \psi), \tag{30}$$

if and only if the triple $(q, Y, Z) \in \mathscr{Q}$ obtained by solving the decoupled FBSDE in ($\mathrm{Opt_N}$) satisfies the optimality condition in ($\mathrm{Opt_N}$). Theorem 25 also guarantees that the maximizer of the problem (30) is unique.

*Step 4: Central planner's problem.* Let $(q, \psi) \in \mathcal{Q}_{c_1} \times \mathcal{A}_{c_2}$ be a saddle point to (P'), for $c_2 > c_2'$. By Step 2, $\psi$ lies in the interior of $\mathcal{A}_{c_2}$. Then Theorem 31 (which is the main result of Subsection 5.3 below) establishes that $\psi \in \mathcal{A}_{c_2}$ is a minimizer of the problem

$$\inf_{\psi \in \mathcal{A}_{c_2}} \mathcal{J}(q, \psi), \tag{31}$$

if and only if the tuple $(\psi, p, k, X) \in \mathscr{A}$ obtained by solving the decoupled FBSDE in ($\mathrm{Opt_C}$) satisfies the optimality condition in ($\mathrm{Opt_C}$). Theorem 31 also guarantees that the minimizer of the problem (31) is unique.

*Step 5: Conclusion.* To conclude the proof, we show that the problems (P') and (P) have the same set of solutions if $c_1 > c_1'$ and $c_2 > c_2'$. We first show that any solution to (P') is solution to (P). To do so, we consider the tuple $(q, Y, Z, \psi, p, k, X) \in \mathscr{Q} \times \mathscr{A}$, solution to the coupled system of FBSDEs ($\mathrm{Opt_C}$)-($\mathrm{Opt_N}$) (which solution is given by the previous two steps). By the sufficiency property of the two first order conditions ($\mathrm{Opt_C}$) and ($\mathrm{Opt_N}$) (see again the previous two steps), $(q, Y, Z, \psi, p, k, X)$ satisfies the following two properties:

- for any $c_1'' > c_1'$, $q$ is a maximizer of (30), with $c_1$ being replaced by $c_1''$ therein, and thus $\mathcal{J}(q, \psi) \geq \mathcal{J}(q', \psi)$ for any $q' \in \mathcal{Q}$;
- for any $c_2'' > c_2'$, $\psi$ is a minimizer of (31), with $c_2$ being replaced by $c_2''$ therein, and thus $\mathcal{J}(q, \psi) \leq \mathcal{J}(q, \psi')$ for any $\psi' \in \mathcal{A}$.

Therefore, $(q, \psi)$ is also a solution to the problem (P), which proves in particular that the problem (P) admits at least a solution.

We now show that any solution to (P) is also a solution to (P'), when $c_1 > c_1'$ and $c_2 > c_2'$. Any solution $(q, \psi) \in \mathcal{Q} \times \mathcal{A}$ to (P) necessarily belongs to $\mathcal{Q}_{c_1''} \times \mathcal{A}_{c_2''}$ for some $c_1'' > 0$ and $c_2'' > 0$ (since $\mathcal{Q} = \cup_{c_1''>0} \mathcal{Q}_{c_1''}$ and $\mathcal{A}^\vartheta = \cup_{c_2''>0} \mathcal{A}^\vartheta_{c_2''}$). This implies that $(q, \psi)$ also lies in $\mathcal{Q}_{c_1'} \times \mathcal{A}_{c_2'}$ by the same argument as in Step 2. Then, by repeating the arguments of Step 3 and 4, we deduce that $(q, \psi)$ is a solution to (P'), concluding the proof.

Uniqueness follows readily. Suppose that there exist two distinct saddle points $(q, \psi)$ and $(q', \psi')$ in $\mathcal{Q} \times \mathcal{A}$, and hence in $\mathcal{Q}_{c_1'} \times \mathcal{A}_{c_2'}$ by the analysis above. Then at least one of the following holds: $q' \neq q$ or $\psi' \neq \psi$. If $q' \neq q$, we use the fact that the optimization problem (30) admits a unique maximizer to deduce that $\mathcal{J}(q, \psi) > \mathcal{J}(q', \psi)$. By the saddle-point property of $(q', \psi')$, this implies

$$\mathcal{J}(q, \psi) > \mathcal{J}(q', \psi) \geq \mathcal{J}(q', \psi').$$

This is a contradiction, since both extreme terms are equal to $\min_{\tilde{\psi} \in \mathcal{A}} \max_{\tilde{q} \in \mathcal{Q}} \mathcal{J}(\tilde{q}, \tilde{\psi})$. Similarly, if $\psi \neq \psi'$, then $\mathcal{J}(q, \psi) < \mathcal{J}(q, \psi') \leq \mathcal{J}(q', \psi')$, which again leads to a contradiction by the saddle-point property. This concludes the proof of uniqueness. □

The system described by equations ($\text{Opt}_\text{C}$) and ($\text{Opt}_\text{N}$) is inherently coupled. Specifically, the control $q$ played by Nature appears both in the pre-Hamiltonian $H$ and in the terminal condition for the adjoint variables $(p, k)$ of the central planner, as shown in ($\text{Opt}_\text{C}$). Similarly, the control $\psi$ of the central planner is present in the driver and in the terminal condition for the Nature adjoint variables $(Y, Z)$. As a consequence of Theorem 10, this coupled system has a unique solution, which characterizes the (unique) saddle point to the problem (P).

*Remark* 11. *Isaac's condition.* At optimality, the following Isaac's condition holds at the optimum:

$$\begin{aligned} &\min_{\psi} \max_{(Y^\star, Z^\star)} \mathcal{H}(t, q_t, Y^\star, Z^\star, Y_t, Z_t, X_t, \psi, p_t, k_t) \\ &= \max_{(Y^\star, Z^\star)} \min_{\psi} \mathcal{H}(t, q_t, Y^\star, Z^\star, Y_t, Z_t, X_t, \psi, p_t, k_t), \end{aligned} \tag{32}$$

$\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$-almost surely. This follows from the combination of ($\text{Opt}_\text{N}$) and ($\text{Opt}_\text{C}$), which say that, $\mathrm{d}\mathbb{P} \otimes \mathrm{d}t$-almost surely,

$$\begin{aligned} (Y_t^\star, Z_t^\star) &\in \arg\max_{(Y^{\star\prime}, Z^{\star\prime})} F(t, q_t, Y^{\star\prime}, Z^{\star\prime}, Y_t, Z_t, \psi_t), \\ \psi_t &\in \arg\min_{\alpha} H(t, X_t, \alpha, p_t, k_t, q_t). \end{aligned}$$

Returning back to the definition (19) of $\mathcal{H}$, these two lines can be rewritten as

$$\begin{aligned} (Y_t^\star, Z_t^\star) &\in \arg\max_{(Y^{\star\prime}, Z^{\star\prime})} \mathcal{H}(t, q_t, Y^{\star\prime}, Z^{\star\prime}, Y_t, Z_t, X_t, \psi_t, p_t, k_t), \\ \psi_t &\in \arg\min_{\alpha} \mathcal{H}(t, q_t, Y_t^\star, Z_t^\star, Y_t, Z_t, X_t, \alpha, p_t, k_t), \end{aligned}$$

from which the bound $\min\max \leq \max\min$ in (32) indeed follows, the converse bound being always true.

### 3.2 Examples of applications

We provide two examples of applications of Theorem 10. On purpose, the presentation is informal and contains no mathematical statement. Further examples are given in Subsection 4.2.

**Risk averse portfolio management with trading costs.** The first example is inspired by [58] and considers the regime "without investment control constraints," in the absence of a risk-free asset, and over a finite time horizon.

Consider a financial market consisting of $d \in \mathbb{N}^\star$ stocks, whose prices per share are encoded in the form of an $d$-dimensional process $S = (S^1, S^2, \dots, S^d)$, satisfying the following SDE:

$$\frac{\mathrm{d}S_t^i}{S_t^i} = c_t^i \mathrm{d}t + (\sigma_t \mathrm{d}W_t)^i, \quad S_0^i = \zeta^i, \quad \forall i \in \{1, \dots, d\},$$

where $\zeta = (\zeta^1, \dots, \zeta^d) \in L^\infty(\mathcal{F}_0, \mathbb{R}^d)$ are the initial prices, $c = (c^1, \dots, c^d) \in L^\infty(\mathbb{F}; \mathbb{R}^d)$ is the vector of stock appreciation rates, $\sigma \in L^\infty(\mathbb{F}, \mathbb{R}^{d\times d})$ is the volatility matrix, and $W = (W^1, \dots, W^d)$ is a $d$-dimensional Brownian motion. For simplicity we thus assume that the number of assets is equal to the number of noise sources. We further assume that there is no bond available on the market. For a given vector of amounts (or allocation strategies) $\psi \in \mathcal{A}$, the dynamics of the self-financing portfolio $X^\psi$ is given by

$$\mathrm{d}X_t = \psi_t \cdot \frac{\mathrm{d}S_t}{S_t}, \quad X_0 = 1,$$

where the dot appearing on the right-hand side stands for the inner product in $\mathbb{R}^d$, the initial condition is arbitrarily chosen to be unitary, and (consistently with the fact there is no bond) the interest rate of the market is assumed to be null for simplicity. The problem of the risk averse investor under a min-max form is given by

$$\sup_{q\in\mathcal{Q}} \inf_{\psi\in\mathcal{A}} \mathcal{J}(q, \psi), \tag{33}$$

where

$$\mathcal{J}(q, \psi) = \mathbb{E}^{\mathbb{Q}}\left[X_T^\psi + \frac{1}{2}\int_0^T |\psi_s|^2 \mathrm{d}s\right] - \lambda \mathrm{H}(\mathbb{Q}|\mathbb{P}), \quad \mathbb{Q} = q_T\mathbb{P}$$

and $q_T = \mathcal{E}_T(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s)$. The problem can be interpreted as follows. Given a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, the investor optimizes the average return of the portfolio while incurring a trading cost. Given an investment strategy chosen by the investor, Nature then selects the worst-case probability measure $\mathbb{Q}$, while being penalized by an entropic cost. The parameter $\lambda > 0$ models the level of risk aversion of the investor.

This is a sub-case of our setting. The random processes $a$ and $b$ are null, $c$ is valued in $\mathbb{R}^d$ instead of $\mathbb{R}^{1\times d}$ and $\sigma$ in $\mathbb{R}^{d\times d}$ instead of $\mathbb{R}^{d\times 1\times d}$. The terminal cost is linear in the measure $\mathbb{Q} \circ (X_T^\psi)^{-1}$ and the actualization rate $Y^\star$ is null. By Theorem 10, the problem (33) admits a unique solution $(\psi, q)$, characterized by $\nabla_\psi H(t, X_t, \psi_t, p_t, k_t, q_t) = 0$ and $Z_t^\star = \partial_z f(Z_t)$, which, after computations (with $k_t$ being viewed as a vector of dimension $d$), gives

$$\psi_t = -q_t^{-1}\left(p_t c_t + \sigma_t^\top k_t\right), \quad Z_t = \frac{1}{\lambda} Z_t^\star,$$

where the tuple of state and adjoint processes $(Y, Z, X, p, k, q)$ is the solution to

$$\begin{cases} -\mathrm{d}Y_t &= (\frac{1}{2\lambda}|Z_t|^2 + |\psi_t|^2)\mathrm{d}t - Z_t \cdot \mathrm{d}W_t, \quad Y_T = X_T, \\ \mathrm{d}X_t &= \psi_t \cdot c_t \mathrm{d}t + \psi_t \cdot (\sigma_t \mathrm{d}W_t), \qquad X_0 = 1, \\ -\mathrm{d}p_t &= -k_t \cdot \mathrm{d}W_t, \qquad p_T = q_T, \\ \mathrm{d}q_t &= \lambda q_t Z_t \cdot \mathrm{d}W_t, \qquad q_0 = 1. \end{cases}$$

By the last two equations, we have that $p_t = q_t$ and $k_t = \lambda q_t Z_t$. Then the solution simplifies to

$$\psi_t = -c_t - \frac{1}{\lambda}\sigma_t^\top Z_t,$$

and

$$\begin{cases} -\mathrm{d}Y_t &= (\frac{1}{2\lambda}|Z_t|^2 + |\psi_t|^2)\mathrm{d}t - Z_t \cdot \mathrm{d}W_t, \quad Y_T = X_T, \\ \mathrm{d}X_t &= \psi_t \cdot c_t \mathrm{d}t + \psi_t \cdot (\sigma_t \mathrm{d}W_t)\,, \qquad X_0 = 1, \end{cases} \tag{34}$$

which reduces the problem to a quadratic FBSDE. It seems that, due to the unboundedness of the terminal condition, the latter system is out of the scope of the theory of FBSDEs with a quadratic driver (in the backward equation) [60, 78, 68]. Very briefly, existing results on the solvability of quadratic BSDEs require the terminal state variable $X_T$ and the cost $\int_0^T |\psi_t|^2 \mathrm{d}t$ (with $\psi$ standing for the optimal control) to admit an exponential moment with a sufficiently large exponent, depending on the parameters of the problem. In the present setting, we are only able to establish exponential integrability for small exponents. This result is not stated explicitly in the article, as it holds only in the case where the function $f$ is genuinely quadratic. In that case, the functional $\mathcal{S}$ coincides with the standard entropy and, by a Donsker–Varadhan-type duality (see (15)), bounds on the conjugate functional $\mathcal{S}^\star$ yield bounds on certain exponential moments of $X_T$.

Here, the solution $(Y, Z)$ to the quadratic BSDE is obtained in the rather weak space

$$D(\mathbb{F}, \mathbb{Q}) \times \bigcap_{\beta \in (0,1)} M^\beta(\mathbb{F}, \mathbb{R}^d, \mathbb{Q}).$$

Because the process $Y$ does not enjoy strong integrability properties, we are not able to justify the following identity, which is frequently used to establish the connection between the min–max and the risk-averse formulations:

$$\frac{1}{\lambda} \ln \mathbb{E}\big[\exp(\lambda Y_0)\big] = \rho_\lambda \left[ X_T + \frac{1}{2} \int_0^T |\psi_s|^2 \, \mathrm{d}s \right],$$

where $\rho_\lambda$ is defined in (11). The standard proof of this identity relies on the Hopf–Cole transform for quadratic BSDEs. However, it would require the random variable

$$X_T + \tfrac{1}{2} \int_0^T |\psi_s|^2 \, \mathrm{d}s$$

to admit an exponential moment of exponent $\lambda$, a property which appears to be out of reach in our framework.

**Control of systemic risk measure.** Systemic risk measures are risk assessment tools that evaluate the macro-level risk of a system composed of multiple interacting agents. The concept was first introduced axiomatically in [38] and has since been extensively explored in both management science [5, 13] and mathematical finance literature [71].

Denoting by $N$ the number of agents in the system, we consider the product space $(\Omega^{\times N}, \mathcal{F}^{\times N}, \mathbb{P}^N := \mathbb{P}^{\times N})$, and we equip its $i$-th factor with an $\mathbb{R}^d$-valued Brownian motion $W^i = (W^i_t)_{t\in[0,T]}$. We denote by $\mathbb{F}^N = (\mathcal{F}^N_t)_{t\in[0,T]}$ the completion of the filtration generated by $(W^1, \dots, W^N)$, and by $\mathcal{Q}^{(N)}$ the analogue of $\mathcal{Q}$ but on the product space, i.e. $\mathcal{Q}^{(N)}$ is the set of $q^N \in L\log L(\mathbb{F}^N)$ such that

$$\mathrm{d}q^N_t = q^N_t \left( \sum_{i=1}^N Z^{\star,i}_t \cdot \mathrm{d}W^i_t \right), \quad q_0 = 1, \quad \mathrm{H}(q^N \mathbb{P}^N | \mathbb{P}^N) < +\infty. \tag{35}$$

Below, we write $\mathbb{Q}^N := q^N \mathbb{P}^N = q^N \mathbb{P}^{\times N}$.

The function $f^\star$ only depends on its last variable and is given, for a certain $\lambda > 0$, by $f^\star(z^{(N)}) = \frac{\lambda}{2} \sum_{i=1}^N |z^i|^2$ for any $z^{(N)} = (z^1, \dots, z^N) \in [\mathbb{R}^d]^N$. We then denote by $\mathcal{A}^{(N)}$ the analogue of $\mathcal{A}$ but on the product space.

To simplify the presentation, we assume that the states of the agents follow dynamics similar to the one presented in the first example, but with each driven by its own noise $W^i$. We also assume that the coefficients are deterministic, which avoids the need to track how each player's coefficients depend on the various sources of noise. For $i \in \{1, \dots, N\}$, the state of the $i$-th agent is thus given by the solution $X^{i,\psi^i}$ of the state equation:

$$\mathrm{d}X^{i,\psi^i}_t = \psi^i_t \cdot c_t \mathrm{d}t + \psi^i_t \cdot \sigma_t \mathrm{d}W^i_t, \quad X_0 = 1, \tag{36}$$

where $\psi^i$ is the control to player $i$.

We now address the construction of a risk measure for the system formed by the $N$ agents. An initial approach would consist in summing individual risk measures associated to each of the agents. However, as emphasized in [13, Section 2], this approach may fail to capture systemic risk effects in financial systems. Motivated by the latter article, we propose an alternative construction in which individual states are first aggregated through an increasing, convex, and nonlinear function $g \colon \mathbb{R} \to \mathbb{R}$, then summed, and finally evaluated via an individual risk measure, the nonlinearity of the aggregation function being essential for practical relevance. A typical example for $g$ is the cost function $g(x) = \max\{x - x_0, 0\}$, where $x_0 \in \mathbb{R}$ is a finite threshold. That said, in order to fit within our framework, in which $g$ is typically required to be differentiable, we consider instead a smooth version of it, sill convex, obtained for instance via regularization. We thus define the systemic risk measure $\mathfrak{p}_\lambda$, by letting

$$\mathfrak{p}^N_\lambda(\psi^1, \dots, \psi^N) = \rho_\lambda \left[ \sum_{i=1}^N \left( g(X^{i,\psi^i}_T) + \frac{1}{2} \int_0^T |\psi^i_t|^2 \mathrm{d}t \right) \right]. \tag{37}$$

*Choice of the normalization.* We emphasize that the sum inside $\rho_\lambda$ in the definition of $\mathfrak{p}^N_\lambda$ diverges as $N$ tends to $+\infty$. In contrast, if we normalize this sum (inside the risk measure) by an additional factor $1/N$, we obtain, in the limit $N \to +\infty$, a model in which risk aversion disappears. To see this, assume that the independence property of the noises $W^1, \dots, W^N$ is asymptotically transmitted to the optimal controls, as in a standard MFC problem without risk aversion. Equivalently, restrict the definition of $\mathfrak{p}_\lambda^N$ to controls $\psi^1, \dots, \psi^N$ that are each constructed as a common progressively measurable function of $W^i$, for the corresponding index $i \in \{1, \dots, N\}$. Then, a purely formal application of the law of large numbers (without further justification) allows one to pass to the limit *inside* $\rho_\lambda$, yielding

$$\lim_{N \to +\infty} \rho_\lambda \left[ \frac{1}{N} \sum_{i=1}^N \left( g(X_T^{i,\psi^i}) + \int_0^T |\psi_t^i|^2 \mathrm{d}t \right) \right] = \int g \mathrm{d}\mu_T^\psi + \rho_\lambda \left[ \int_0^T |\psi_t|^2 \mathrm{d}t \right], \quad (38)$$

where $\mu_T^\psi = \mathcal{L}(X_T^\psi)$ and the dynamics of $X^\psi$ is given by

$$\mathrm{d}X_t = \psi_t \cdot c_t \mathrm{d}t + \psi_t \cdot (\sigma_t \mathrm{d}W_t), \quad X_0 = 1. \quad (39)$$

As announced, the risk aversion has disappeared in the limit. Intuitively, the limiting problem (obtained by letting $N \to +\infty$) is a standard MFC problem. For this reason, we propose below an alternative construction of risk measures for $N$-particles system.

*Solving the $N$-fixed problem.* Instead, we want to keep the sum over $g(X_T^{i,\psi})$ unnormalized in the definition (37) of $p_\lambda^N$. Accordingly, our objective is to explain, at least informally, what is the behaviour of $\frac{1}{N} p_\lambda^N$ as $N$ tends to $+\infty$.

The first step is to observe from the Donsker-Varadhan formula (15) that the minimization of $\mathfrak{p}_\lambda^N$ can be reformulated as as a min-max problem, i.e.,

$$\begin{aligned} &\inf_{\psi^N \in \mathcal{A}^{(N)}} \tfrac{1}{N} \mathfrak{p}_\lambda^N(\psi^1, \dots, \psi^N) \\ &= \inf_{\psi^{(N)} \in \mathcal{A}^{(N)}} \sup_{q^N \in \mathcal{Q}^{(N)}} \frac{1}{N} \left\{ \mathbb{E}^{\mathbb{Q}^N} \left[ \sum_{i=1}^N \left( g\left(X_T^{i,\psi^i}\right) + \int_0^T |\psi_t^i|^2 \mathrm{d}t \right) \right] - \lambda \mathrm{H}\left(\mathbb{Q}^N | \mathbb{P}^N\right) \right\}. \end{aligned}$$

As $g$ is convex, we observe that the min-max problem appearing in the right-hand side enters the framework of Theorem 10, with $n = N$ and $d$ replaced by $d \times N$, with $q^N$ and $Z^{\star,(N)} = (Z^{\star,i})_{i \in \{1,\dots,N\}}$ satisfying (35) (implicitly, $\alpha = 0$ and $Y^{\star,(N)} = (Y^{\star,i})_{i \in \{1,\dots,N\}} \equiv 0$), with $\psi^{(N)} = (\psi^i)_{i \in \{1,\dots,N\}}$ and $X^{(N)} = (X^{i,\psi^i})_{i \in \{1,\dots,N\}}$ solving (36), and with the terminal cost functions

$$\mathcal{G}\left(q_T^N, X_T^{(N)}\right) = \mathbb{E}\left[ q_T^N \sum_{i=1}^N g\left(X_T^{i,\psi^i}\right) \right],$$

$$\ell\left(t, \psi^{(N)}\right) = \frac{1}{2} \sum_{i=1}^N |\psi_t^i|^2, \ f^\star\left(t, Z^{\star,(N)}\right) = \frac{\lambda}{2} \sum_{i=1}^N |Z_t^{\star,i}|^2.$$

The saddle-point is characterized by

$$\psi_t^i = -(q_t^N)^{-1} \left( p_t^i c_t + \sigma_t^\top k_t^i \right), \quad Z_t^i = \frac{1}{\lambda} Z_t^{\star,i},$$

for each $i \in \{1, \dots, N\}$, where $p_t^i$ takes values in $\mathbb{R}$ and $k_t^i$ in $\mathbb{R}^d$, and where the tuple of state and adjoint processes $(Y^N, Z^{(N)} = (Z^1, \dots, Z^N), X^{(N)}, p^{(N)} = (p^1, \cdots, p^N), k^{(N)} = (k^1, \dots, k^N), q^N)$ is the solution to

$$\begin{cases} -\mathrm{d}Y_t^N &= \left( \frac{1}{2\lambda} \sum_{j=1}^N |Z_t^j|^2 + \frac{1}{2} \sum_{j=1}^N |\psi_t^j|^2 \right) \mathrm{d}t - \sum_{j=1}^N Z_t^j \cdot \mathrm{d}W_t^j, \\ \mathrm{d}X_t^i &= \psi_t^i \cdot c_t \mathrm{d}t + \psi_t^i \cdot (\sigma_t \mathrm{d}W_t^i), \\ -\mathrm{d}p_t^i &= -\sum_{j=1}^N k_t^{i,j} \cdot \mathrm{d}W_t^j, \\ \mathrm{d}q_t^N &= \lambda q_t^N \sum_{j=1}^N Z_t^j \cdot \mathrm{d}W_t^j, \end{cases}$$

with terminal conditions

$$Y_T^N = \sum_{j=1}^N g(X_T^{j,\psi^j}), \quad X_0^{N,i} = 1, \quad p_T^i = q_T^N g'\left(X_T^{i,\psi^i}\right), \quad q_0^N = 1,$$

for any $i \in \{1, \dots, N\}$.

*Towards a robust MFC problem.* We now provide a heuristic derivation of the limiting problem as $N \to +\infty$. The purpose of this discussion is solely to identify the structure of the limiting model; no claim of rigor is made at this stage. Proceeding as in (38), we assume that, at the saddle point, the controls $\psi^1, \dots, \psi^N$ are each constructed as a common progressively measurable function of the individual Brownian motion $W^i$, for the corresponding index $i \in \{1, \dots, N\}$, and similarly for the controls $Z^{\star,1}, \dots, Z^{\star,N}$. Strictly speaking, such an independence structure does not hold at the finite-$N$ saddle point. However, this assumption can be justified a posteriori by reverse engineering: starting from a solution to the limiting problem, one may construct an approximate optimizer for the finite-$N$ problem, which is a standard approach in mean field control theory. Under this assumption, and applying Girsanov's theorem (all similar changes of probability measures will be justified in the core of the article, but we prefer not to address such technical questions in this informal discussion), the vector $X^{(N)} = (X^1, \dots, X^N)$ satisfies the following dynamics under $\mathbb{Q}^N$:

$$\mathrm{d}X_t^i = \psi_t^i \cdot \left(c_t + \sigma_t Z_t^{\star,i}\right) \mathrm{d}t + \psi_t^i \cdot \sigma_t \, \mathrm{d}W_t^{i,\mathbb{Q}^N},$$

where

$$\left(W_t^{i,\mathbb{Q}^N} = W_t^i - \int_0^t Z_s^{\star,i} \, \mathrm{d}s\right)_{t\in[0,T],\, i=1,\dots,N}$$

is (expected to be) an $N$-dimensional Brownian motion under $\mathbb{Q}^N$.

And then, under $\mathbb{Q}^N$, the processes $(X^i, \psi^i)_{i=1,\cdots,N}$ are independent and identically distributed, which makes it possible to derive, by a new application of the law of large numbers, the following approximation for the cost underpinning the min-max problem:

$$\begin{aligned}
&\frac{1}{N}\left\{\mathbb{E}^{\mathbb{Q}^N}\left[\sum_{i=1}^N \left(g\left(X_T^{i,\psi^N}\right) + \int_0^T |\psi_t^{N,i}|^2 \mathrm{d}t\right)\right] - \lambda \mathrm{H}\left(\mathbb{Q}^N | \mathbb{P}^N\right)\right\} \\
&\approx G\left((q_T^1 \mathbb{P})_{X_T^{1,\psi^1}}\right) + \mathbb{E}\left[q_T^1 \int_0^T |\psi_t^1|^2 \mathrm{d}t\right] - \lambda \mathrm{H}\left(q_T^1 \mathbb{P} | \mathbb{P}\right).
\end{aligned}$$

as $N \to +\infty$, where $q_T^1\mathbb{P})_{X_T^{1,\psi^1}}$ denotes the law of $X_T^{1,\psi^1}$, regarded as a random variable on $(\Omega, \mathcal{F})$ equipped with the probability measure $q_T^1\mathbb{P}$.

We thus conjecture that, in the limit, we are faced with a robust control problem involving a single agent, but with a cost depending on the law of the agent under the distribution resulting from Nature's choice. In other words, we expect that the asymptotic problem (obtained by letting $N \to \infty$) consists of the following min–max problem:

$$\inf_{\psi\in\mathcal{A}} \sup_{q\in\mathcal{Q}} \left\{ G\big(\mathcal{L}^{\mathbb{Q}}(X_T^{\psi})\big) + \mathbb{E}^{\mathbb{Q}}\left[\int_0^T |\psi_t|^2 \, \mathrm{d}t\right] - \lambda \, \mathrm{H}\big(\mathbb{Q}|\mathbb{P}\big) \right\},$$

formulated on $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{Q} := q_T \mathbb{P}$ denotes the probability measure induced by Nature's strategy. A more general treatment of this problem is provided in the next section, which is devoted to the mean field regime.

# 4 Applications to mean field models

In this section, we develop a robust formulation of classical mean field control problems and subsequently analyze an associated variational mean field game problem. Subsection 4.1 is devoted to the robust mean field control problem (MFC). Relying on Theorem 10 from the previous section, we establish in Corollary 15 the existence and uniqueness of a saddle point, together with the corresponding stochastic maximum principle. Subsection 4.2 provides two examples of applications in this context. Subsection 4.3 then turns to a class of variational mean field games. Building on Corollary 15, we prove in Corollary 16 the existence and uniqueness of a Nash equilibrium, which is fully characterized by a McKean–Vlasov forward–backward stochastic differential equation.

## 4.1 Robust mean field control

This subsection is dedicated to the study of the robust mean field control problem (MFC), which we recall here

$$\inf_{\psi \in \mathcal{A}} \sup_{q \in \mathcal{Q}} \left\{ G\left( (q_T \mathbb{P})_{X_T^\psi} \right) + \mathbb{E}\left[ \int_0^T q_s \ell(s, \psi_s) \mathrm{d}s \right] - \mathcal{S}(q) \right\}, \tag{MFC}$$

where $G\colon \mathcal{M}_+(\mathbb{R}^n) \to \mathbb{R}$ is a mean field mapping of the positive measure $(q_T\mathbb{P})_{X_T^\psi} = (q_T\mathbb{P}) \circ X_T^{\psi,-1}$ and $(X_t^\psi)_{t\in[0,T]}$ denotes the solution to the controlled stochastic differential equation (7).

This problem can be recast as problem (P) assuming that the mapping $\mathcal{G}$ is specified as follows

$$\mathcal{G}(q, X) = G\left((q\mathbb{P})_X\right). \tag{40}$$

When the actualization rate $Y^\star$ is null, the measure $q_T\mathbb{P}$ is a probability measure, equal to $\mathcal{E}_T(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s)$, and the domain of definition of $G$ can be reduced to $\mathcal{P}(\mathbb{R}^n)$. In the latter case, $G$ is a true mean field function. By extension, we still call the model 'mean field' even if the mass of $q_T$ is unnormalized.

As the domain of definition of $G$ is larger than $\mathcal{P}_1(\mathbb{R}^n)$, we are led, under the assumptions below, to redefine implicitly the notion of flat and Lions derivatives. Since the objects thus redefined coincide, in the mean field case, with the true flat and Lions derivatives, we nevertheless use the same notations $\delta G/\delta\mu$ and $\partial_\mu G$ as in the introduction. This is the rational behind the introduction of the following distances.

**Spaces of positive measures** We introduce, for any $p \geq 1$, a variant of the total variation distance, adapted to elements of $\mathcal{M}_p(\mathbb{R}^n) := \{\mu \in \mathcal{M}_+(\mathbb{R}^n),\ M_p(\mu) < +\infty\}$, where here and throughout $M_p(\mu) := \int_{\mathbb{R}^n} |x|^p \mathrm{d}\mu(x)$. For such a $p$, we let (the proof of the fact that the right-hand side below defines a distance is left to the reader):

$$d_p(\mu,\nu) := \sup_{\varphi} \int_{\mathbb{R}^n} \varphi(x)\mathrm{d}\left(\mu-\nu\right)(x), \quad \mu,\nu \in \mathcal{M}_p(\mathbb{R}^n), \tag{41}$$

where the supremum is taken over measurable functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $|\varphi(x)| \leq 1 + |x|^p$.

We also use the standard $p$-Wasserstein distance, when restricted to subsets of measures with equal mass. Below, we refer to these subsets as "isomass subsets". For the same $p$ as above, and for $\mu$ and $\nu$ in $\mathcal{M}_p(\mathbb{R}^n)$ such that $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n)$, we let

$$W_p(\mu,\nu) := \inf_{\substack{\pi\in\mathcal{M}_p(\mathbb{R}^n\times\mathbb{R}^n),\\ \pi\circ e_1^{-1}=\mu,\ \pi\circ e_2^{-1}=\nu}} \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-x'|^p \mathrm{d}\pi(x,x') \leq \upsilon,$$

where $e_i : \mathbb{R}^n \times \mathbb{R}^n \ni (x_1,x_2) \mapsto x_i$, for $i = 1,2$.

We combine the two distances $d_p$ and $W_p$ by considering functions, defined on $\mathcal{M}_p(\mathbb{R}^n)$, that are continuous with respect to $d_p$ on the entire $\mathcal{M}_p(\mathbb{R}^n)$, and that are continuous with respect to $W_p$ on any isomass subset of $\mathcal{M}_p(\mathbb{R}^n)$. We prove in Subsection D.1 of the Appendix that those functions are continuous with respect to the so-called generalized $p$-Wasserstein distance, and conversely. That said, we feel easier, in our specific framework, to use separately the two distances $d_p$ and $W_p$, instead of the single generalized Wasserstein distance.

**Assumptions** We now state the required assumptions on $G$. We still assume A1-A5 to hold. We recall that, the parameter $r$ used throughout, is defined in A2.

A9 We assume that there exist two functions

$$\frac{\delta G}{\delta\mu} : \mathcal{M}_{2-r}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}, \quad \partial_\mu G : \mathcal{M}_{2-r}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n,$$

with $\delta G/\delta\mu$ being differentiable in the second argument when the first one is fixed such that, for any $\mu \in \mathcal{M}_{2-r}(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$,

$$\frac{\delta G}{\delta\mu}(\mu,x) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0+} G\left(\mu+\varepsilon\delta_x\right), \tag{42}$$

where $\delta_x$ is the delta mass at point $x$, and

$$\partial_\mu G(\mu,x) = \nabla_x \frac{\delta}{\delta\mu} G(\mu,x). \tag{43}$$

a) We assume that these three mappings satisfy the following growth conditions:

$$\begin{aligned} |G(\mu)| &\leq L\left(1 + M_{2-r}(\mu)\right), \\ -L\left(1+M_{2-r}(\mu)+|x|\right) \leq \frac{\delta G}{\delta\mu}(\mu)(x) &\leq L\left(1+M_{2-r}(\mu)+|x|^{2-r}\right), \\ |\partial_\mu G(\mu,x)| &\leq L\left(1+M_{2-r}(\mu)+|x|^{1-r}\right). \end{aligned} \tag{44}$$

b) We assume that $G$ is continuous with respect to $d_{2-r}$ on the entire $\mathcal{M}_{2-r}(\mathbb{R}^n)$, and continuous with respect to $W_{2-r}$ on isomass subsets.

As for the derivative $\delta G/\delta\mu$ (which is already required to be locally Lipschitz, uniformly in $\mu$, thanks to (44)), we assume it to be continuous with respect to $d_{2-r}$, with a modulus of continuity that grows at most like $|x|^{2-r}$, and that is uniform in $\mu \in \mathcal{M}_{2-r}(\mathbb{R}^n)$ satisfying $M_{2-r}(\mu) \leq C$ for some $C > 0$. More precisely, for any $\varepsilon > 0$ and $C > 0$, we assume that there exists $\upsilon > 0$, such that, for any $\mu, \mu' \in \mathcal{M}_{2-r}(\mathbb{R}^n)$ satisfying $M_{2-r}(\mu), M_{2-r}(\mu') \leq C$ and $d_{2-r}(\mu, \mu') \leq \upsilon$, we have

$$\sup_{x\in\mathbb{R}^n} \left[ \frac{1}{1+|x|^{2-r}} \left| \frac{\delta G}{\delta\mu}(\mu)(x) - \frac{\delta G}{\delta\mu}(\mu')(x) \right| \right] \leq \varepsilon. \tag{45}$$

We also assume that, for any $x \in \mathbb{R}^n$, $\mu \mapsto [\delta G/\delta\mu](\mu, x)$ is continuous with respect to $W_{2-r}$ on isomass subsets.

At last, we require that, when the first argument is restricted to the isomass subset $\{\mu \in \mathcal{M}_{2-r}(\mathbb{R}^n), \mu(\mathbb{R}^n) = c\}$, for some $c \geq 0$, the function $\partial_\mu G$ is locally Lipschitz continuous in $(\mu, x)$ in the following sense: for any $C > 0$, there exists $L_C \geq 0$ such that, for any $\mu, \mu' \in \mathcal{M}_{2-r}(\mathbb{R}^n)$ with $\mu(\mathbb{R}^n) = \mu'(\mathbb{R}^n)$ and $M_{2-r}(\mu), M_{2-r}(\mu') \leq C$, and any $x, x' \in \mathbb{R}^n$,

$$\begin{aligned} \frac{1}{1+|x|^{1-r}} \left| \partial_\mu G(\mu, x) - \partial_\mu G(\mu', x) \right| &\leq L_C W_{2-r}(\mu, \mu'), \\ \left| \partial_\mu G(\mu, x) - \partial_\mu G(\mu, x') \right| &\leq L_C |x - x'|. \end{aligned} \tag{46}$$

c) We finally assume that $G$ is flat concave, i.e., for $\mu, \mu' \in \mathcal{M}_{2-r}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \left( \frac{\delta G}{\delta\mu}(\mu, x) - \frac{\delta G}{\delta\mu}(\mu', x) \right) \mathrm{d}(\mu - \mu')(x) \leq 0, \tag{47}$$

and $G$ is displacement convex on isomass subsets, i.e., for any $\mu, \mu' \in \mathcal{M}_{2-r}(\mathbb{R}^n)$ with $\mu(\mathbb{R}^n) = \mu'(\mathbb{R}^n)$, for any measure $\pi \in \mathcal{M}_{2-r}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\mu$ and $\mu'$ as marginals,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \partial_\mu G(\mu, x) - \partial_\mu G(\mu', x') \right) \cdot (x - x') \mathrm{d}\pi(x, x') \geq 0. \tag{48}$$

*Remark* 12. For presentation purpose we only consider a mean field terminal cost. But one could also consider mean field running cost of the separated form

$$\ell'(t, \psi_t, \mu_t) = \ell(t, \psi_t) + c(X_t, \mu_t),$$

where $\mu_t$ is the marginal law of $X_t$ under the probability measure $q_T\mathbb{P}$ induced by Nature. The assumptions on $c$ should be analogous to the assumptions required for $g$ above (growth, flat differentiable and Lions differentiable, with the appropriate regularity, flat concave and displacement convex).

*Remark* 13. The following comments are in order.

The first remark is that it suffices, for our purpose, to have all the above conditions satisfied for $\mu$ and $\mu'$ of mass less than $\exp(\alpha T)$. This follows from the fact that, in our applications, $\mathbb{E}[q_T] \leq \exp(\alpha T)$.

The second observation is that the notion of displacement convexity, as mentioned in (48), is usually reserved to functions defined on the space of probability measures. In (48), we can easily recover the case when $\mu$ and $\mu'$ are probability measures by normalizing them. Indeed, for a given $c > 0$ representing the common mass of $\mu$ and $\mu'$, we can consider the function $G^{(c)} : \mathcal{P}_{2-r}(\mathbb{R}^n) \ni \mu \mapsto G(c\mu)$. Obviously, the standard flat and Lions derivatives (according to their usual definitions for functionals defined on $\mathcal{P}_{2-r}(\mathbb{R}^n)$, the common construction of the Lions derivative being restricted to the case $r = 0$) are

$$\frac{\delta G^{(c)}}{\delta \mu}(\mu, x) = c\frac{\delta G}{\delta \mu}(\mu, x), \quad \partial_\mu G^{(c)}(\mu, x) = c\partial_\mu G(\mu, x).$$

If $G^{(c)}$ has second-order derivatives in $\mu$ and $x$, then it satisfies (48) for any two probability measures $\mu$ and $\mu'$ if

$$\begin{aligned} &\int_{\mathbb{R}^n \times \mathbb{R}^n} \mathrm{Tr}\left[\partial^2_\mu G^{(c)}(\mu, x, x')\beta(x) \otimes \beta(x')\mathrm{d}\mu(x)\right] \mathrm{d}\mu(x') \\ &+ \int_{\mathbb{R}^n} \mathrm{Tr}\left[\partial_x \partial_\mu G^{(c)}(\mu, x)\beta(x) \otimes \beta(x)\right] \mathrm{d}\mu(x) \geq 0, \end{aligned} \tag{49}$$

for any bounded measurable function from $\mathbb{R}^n$ to itself. The above can be found in [35, Chapter 5], when $r = 0$. Returning back to unnormalized measures (i.e., changing $\mu$ into $c\mu$), it easy to see that, when $r = 0$, (48) is true (whathever the mass of $\mu$ and $\mu'$) if (49) is true with $G$ being substituted for $G^{(c)}$. In fact, (49) remains also a sufficient condition when $r = 1$: It implies (48) when $\mu$ and $\mu'$ therein have finite second-order moments; by a standard approximation argument, the inequality remains true when $\mu$ and $\mu'$ are just in $\mathcal{M}_1(\mathbb{R}^n)$. Below, we thus call Hessian of $G$ in the direction $\beta$ the quantity

$$\begin{aligned} \mathcal{H}_G(\beta) := &\int_{\mathbb{R}^n \times \mathbb{R}^n} \mathrm{Tr}\left[\partial^2_\mu G(\mu, x, x')\beta(x) \otimes \beta(x')\right] \mathrm{d}\mu(x)\mathrm{d}\mu(x') \\ &+ \int_{\mathbb{R}^n} \mathrm{Tr}\left[\partial_x \partial_\mu G(\mu, x)\beta(x) \otimes \beta(x)\right] \mathrm{d}\mu(x). \end{aligned}$$

**Constructing flat concave and displacement convex functions** We first note that any linear functional of the form

$$G_1(\mu) = \int_{\mathbb{R}^n} v_1(x)\mathrm{d}\mu(x), \tag{50}$$

where $v_1 : \mathbb{R}^n \to \mathbb{R}$ is smooth and convex, is an ideal candidate to satisfy Assumption A9. Indeed it is flat concave as it is linear in $\mu$ and displacement convex by convexity of $v_1$. The main point is to check that $v_1$ satisfies the required integrability properties, depending on whether $r = 0$ or $r = 1$, which prompts us to distinguish between this two cases below.

Regardless of the integrability properties, $G_1$ satisfies

$$\frac{\delta G_1}{\delta \mu}(\mu, x) = v_1(x),$$

for any $x \in \mathbb{R}^n$, so that (47) is trivially satisfied, and

$$\partial_\mu G_1(\mu, x) = \nabla_x v_1(x),$$

so that (48) is expected to be satisfied if $v_1$ is convex. In particular, the Hessian $\mathcal{H}_{G_1}(\beta)$ is equal to

$$\mathcal{H}_{G_1}(\beta) = \int_{\mathbb{R}^n} \mathrm{Tr}\left[\nabla_x^2 v_1(x)\beta(x)\otimes\beta(x)\right] \mathrm{d}\mu(x), \tag{51}$$

which is obviously non-negative when $v_1$ is convex.

Before we discuss more in depth the integrability properties, we notice, as $G_1$ is linear in $\mu$, that any composition of $G_1$ by a (smooth) concave function $\varphi : \mathbb{R} \to \mathbb{R}$ is expected to be flat concave. Such an example can be written as

$$G_2(\mu) = \varphi\left(\int_{\mathbb{R}^n} v_2(x)\mathrm{d}\mu(x)\right), \tag{52}$$

where $v_2$ satisfies the required integrability constraints (similar to $v_1$, as discussed below), and $\varphi$ is smooth and concave. In this situation, we have (at least formally),

$$\partial_\mu G_2(\mu, x) = \varphi'\left(\int_{\mathbb{R}^n} v_2(x)\mathrm{d}\mu(x)\right)\nabla_x v_2(x),$$

and then,

$$\begin{aligned}
\partial_\mu^2 G_2(\mu, x, x') &= \varphi''\left(\int_{\mathbb{R}^n} v_2(x)\mathrm{d}\mu(x)\right)\nabla_x v_2(x)\otimes\nabla_x v_2(x'),\\
\nabla_x\partial_\mu G_2(\mu, x) &= \varphi'\left(\int_{\mathbb{R}^n} v_2(x)\right)\nabla_x^2 v_2(x).
\end{aligned}$$

In particular, by Cauchy-Schwarz inequality, it is quite easy that, for $\mu(\mathbb{R}^n) = \mu'(\mathbb{R}^n) \leq \exp(\alpha T)$,

$$|\mathcal{H}_{G_2}(\beta)| \leq C(\varphi, v_2, \mu)\int_{\mathbb{R}^n}\left(|\nabla_x v_2(x)|^2 + |\nabla_x^2 v_2(x)|\right)\mathrm{d}\mu(x), \tag{53}$$

where $C(\varphi, v_2, \mu) := \max(|\varphi'|(\int_{\mathbb{R}^n} v_2 \mathrm{d}\mu), \exp(\alpha T)|\varphi''|(\int_{\mathbb{R}^n} v_2\mathrm{d}\mu))$.

To produce a wider class class of functions that are flat concave and displacement convex, we can sum $G_1$ and $G_2$. Indeed, we observe that $G_1 + G_2$ is always flat concave. To obtain that the sum is displacement convex, we only need to ensure that

$$\mathcal{H}_{G_1}(\beta) + \mathcal{H}_{G_2}(\beta) \geq 0.$$

Combining (51) and (53), the latter inequality holds true if

$$\forall x, y \in \mathbb{R}^n, \quad \mathrm{Tr}\left[\nabla_x^2 v_1(x) y\otimes y\right] \geq C(\varphi, v_2, \mu)\left(|\nabla_x v_2|^2 + |\nabla_x^2 v_2(x)|\right)|y|^2. \tag{54}$$

We stress that the inequality must be true for any $\mu$ with a mass less than $\exp(\alpha T)$. This puts an additional constraint due to the dependence of the constant $C(\varphi, v_2, \mu)$ on $\mu$. That said, when $v_2$ is bounded, the constant $C(\varphi, v_2, \mu)$ can be bounded independently of $\mu$, since $\mu(\mathbb{R}^n) \leq \exp(\alpha T)$; in that case, we can substitute $C(\varphi, v_2)$ for $C(\varphi, v_2, \mu)$ and then get a condition that is independent of $\mu$.

In order to give more explicit examples, we need to take into account the integrability conditions of $\mu$, as the latter dictate the growth properties of the derivatives of $v_1$ and $v_2$.

*Case* $r = 0$. When $r = 0$, a prototypal example is $v_1(x) = \lambda|x|^2/2$, for $\lambda > 0$. Then, (54) holds if

$$C(\varphi, v_2, \mu) \left(|\nabla_x v_2|^2 + |\nabla^2_x v_2(x)|\right) \leq \lambda.$$

An interesting example is $v_2 = x$ and $\varphi(u) = u^2/2$, in which case $G = G_1 + G_2$ writes

$$G(\mu) = \frac{\lambda}{2} \int_{\mathbb{R}^n} |x|^2 \mathrm{d}\mu(x) - \frac{1}{2} \left| \int_{\mathbb{R}^n} x \mathrm{d}\mu(x) \right|^2.$$

Here,

$$\frac{\delta G}{\delta \mu}(\mu, x) = \frac{\lambda}{2}|x|^2 - x \cdot \left( \int_{\mathbb{R}^n} x' \mathrm{d}\mu(x') \right), \quad \partial_\mu G(\mu, x) = \lambda x - \int_{\mathbb{R}^n} x' \mathrm{d}\mu(x'),$$

and it is easy to check (44), (45) and (46). Moreover, for any measure $\pi \in \mathcal{M}_2(\mathbb{R}^n \times \mathbb{R}^n)$ with $\mu$ and $\mu'$ as marginal measures,

$$\begin{aligned}
&\int_{\mathbb{R}^n \times \mathbb{R}^n} \left(\partial_\mu G(\mu, x) - \partial_\mu G(\mu', x')\right) \cdot (x - x') \mathrm{d}\pi(x, x') \\
&= \lambda \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - x'|^2 \mathrm{d}\pi(x, x') - \left| \int_{\mathbb{R}^n \times \mathbb{R}^n} (x - x') \mathrm{d}\pi(x, x') \right|^2 \\
&\geq (\lambda - \mu(\mathbb{R}^n)) \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - x'|^2 \mathrm{d}\pi(x, x'),
\end{aligned}$$

with the last line following from Cauchy-Schwarz inequality, and from the fact that $\pi(\mathbb{R}^n \times \mathbb{R}^n) = \mu(\mathbb{R}^n)$. This shows that, for $\lambda \geq \exp(\alpha T)$, (48) is satisfied for any $\mu, \mu'$ such that $\mu(\mathbb{R}^n) = \mu'(\mathbb{R}^n) \leq \exp(\alpha T)$. At the threshold $\lambda = \mu(\mathbb{R}^n) (= \mu'(\mathbb{R}^n))$,

$$G(\mu) = \frac{1}{4} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - x'|^2 \mathrm{d}\mu(x) \mathrm{d}\mu(x').$$

Still for $v_1(x) = \lambda|x|^2/2$, we can choose $v_2$ bounded, with bounded derivatives of order 1, 2 and 3. In that case, it is easy to check (44), (45) and (46). Moreover, (54) holds true if

$$\lambda \geq \sup_{|c| \leq \exp(\alpha T) \|v_2\|_\infty} \max\left(|\varphi'|(c), \exp(\alpha T)|\varphi''|(c)\right) \left(\|\nabla_x v_2\|^2_\infty + \|\nabla^2_x v_2\|_\infty\right).$$

*Case* $r = 1$. When $r = 1$, we can no longer choose $v_1$ of quadratic growth (since $v_1$ must have a finite integral with respect to elements of $\mathcal{M}_1(\mathbb{R}^n)$). Instead, we can work with

$$v_1(x) = \lambda v_1^0(x), \quad \text{with} \quad v_1^0(x) := \left(1 + |x|^2\right)^{1/2},$$

for some $\lambda > 0$. Then, for any coordinates $i, j \in \{1, \dots, n\}^2$,

$$\partial_{x_i} v_1^0(x) = \frac{x_i}{v_1^0(x)}, \quad \partial^2_{x_i x_j} v_1^0(x) = \frac{\delta_{i,j}}{v_1^0(x)} - \frac{x_i x_j}{v_1^0(x)^3},$$

where $\delta$ is the Kronecker delta here, which gives for any $y \in \mathbb{R}^n$,

$$\mathrm{Tr}\left[\nabla^2_x v_1(x) y \otimes y\right] = \lambda \left[\frac{|y|^2}{v_1^0(x)} - \frac{(x \cdot y)^2}{v_1^0(x)^3}\right] \geq \lambda \frac{|y|^2}{v_1^0(x)^3}.$$

If we assume that $v_2$ is bounded, with bounded derivatives of order 1, 2 and 3, it is easy to check (44), (45) and (46). Moreover, (54) holds true if

$$\frac{\lambda}{v_1^0(x)^3} \geq \sup_{|c| \leq \exp(\alpha T) \|v_2\|_\infty} \max\left(|\varphi'|(c), \exp(\alpha T)|\varphi''|(c)\right) \left(|\nabla_x v_2(x)|^2 + |\nabla^2_x v_2(x)|\right).$$

For instance, the above holds true if $v_2$ is compactly supported and $\lambda$ is large enough.

**Main result** The following result is standard in the literature (see for instance [35, 89]). For completeness, the proof is given in the Appendix, see Subsection D.2.

**Lemma 14.** *Let $G$ satisfy A9. Then, on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as before, for any two random variables $q, q'$ with values in $\mathbb{R}_+$, such that $\mathbb{E}[q], \mathbb{E}[q'] < +\infty$, and any random variable $X$ with values in $\mathbb{R}^n$, such that $\mathbb{E}[q|X|^{2-r}]$, $\mathbb{E}[q'|X|^{2-r}] < +\infty$,*

$$\begin{aligned} &G\left((q'\mathbb{P})_X\right) - G\left((q\mathbb{P})_X\right) \\ &\quad = \int_0^1 \left[\int_{\mathbb{R}^n} \frac{\delta G}{\delta \mu}\left(\left((\theta q' + (1-\theta)q)\mathbb{P}\right)_X, x\right) \mathrm{d}\left[(q'\mathbb{P})_X - (q\mathbb{P})_X\right](x)\right] \mathrm{d}\theta, \end{aligned} \tag{55}$$

*and, for any random variable $X'$ with values in $\mathbb{R}^n$, such that $\mathbb{E}[q|X'|^{2-r}] < +\infty$,*

$$\begin{aligned} &G\left((q\mathbb{P})_{X'}\right) - G\left((q\mathbb{P})_X\right) \\ &\quad = \int_0^1 \mathbb{E}\left[\partial_\mu G\left((q\mathbb{P})_{\theta X' + (1-\theta)X}, \theta X' + (1-\theta)X\right) \cdot \left(X' - X\right)\right] \mathrm{d}\theta. \end{aligned} \tag{56}$$

Here is now the main result of this section:

**Corollary 15.** *Let Assumptions A1–A5 and A9 be satisfied. Then, there exists a unique saddle point $(\bar{q}, \bar{\psi}) \in \mathcal{Q} \times \mathcal{A}$ to the problem (MFC). Moreover, if a pair $(q, \psi) \in \mathcal{Q} \times \mathcal{A}$ is a solution to the problem (P), then the tuples $(\psi, p, k, X)$, obtained by solving in $\mathscr{A}$ the two decoupled equations in ($\mathrm{Opt_N}$) with the terminal condition being specified by*

$$p_T = q_T \partial_\mu G\left((q_T\mathbb{P})_{X_T}, X_T\right),$$

*and $(q, Y, Z)$, obtained by solving in $\mathscr{Q}$ the two decoupled equations in ($\mathrm{Opt_C}$) with the terminal condition being specified by*

$$Y_T = \frac{\delta G}{\delta \mu}\left((q_T\mathbb{P})_{X_T}, X_T\right),$$

*satisfy the optimality conditions in ($\mathrm{Opt_N}$) and ($\mathrm{Opt_C}$) respectively. Conversely, if $(\psi, p, k, X, q, Y, Z) \in \mathscr{A} \times \mathscr{Q}$ is a solution to ($\mathrm{Opt_C}$)-($\mathrm{Opt_N}$) with the terminal condition specified above, then the pair $(\psi, q) \in \mathcal{A} \times \mathcal{Q}$ is the unique solution to the problem (MFC).*

*Proof.* As the result is a direct application of Theorem 10, we just need to check that the mapping $\mathcal{G}$ defined in (40) satisfies the Assumptions A6-A8 of the previous section.

*Step 1: $\mathcal{G}$ verifies A6-A7.* Let $(q, X), (q', X') \in \mathscr{G}$ (the definition of $\mathscr{G}$ can be found in Assumption A6), satisfying $\mathbb{E}[q|X'|^{2-r}]$ and $\mathbb{E}[q'|X|^{2-r}] < +\infty$. By (44), we can easily check A6 with

$$\mathcal{G}(q, X) = G\left((q\mathbb{P})_X\right), \ \delta_\mu \mathcal{G}(q, X) = \frac{\delta G}{\delta \mu}\left((q\mathbb{P})_X, X\right), \ \delta_X \mathcal{G}(q, X) = \partial_\mu G\left((q\mathbb{P})_X, X\right).$$

In fact, the main point is to check that $\delta_\mu \mathcal{G}$ and $\delta_X \mathcal{G}$ are the derivatives of $\mathcal{G}$, in the directions $q$ and $X$ respectively, as required in A7. By Lemma 14, we know that

$$
\begin{aligned}
G\left((q'\mathbb{P})_X\right) &= G\left((q\mathbb{P})_X\right) \\
&\quad + \int_0^1 \left[\int_{\mathbb{R}^n} \frac{\delta G}{\delta\mu}\left((q^\theta\mathbb{P})_X, x\right) \mathrm{d}\left[(q'\mathbb{P})_X - (q\mathbb{P})_X\right](x)\right] \mathrm{d}\theta, \\
G\left((q\mathbb{P})_{X'}\right) &= G\left((q\mathbb{P})_X\right) + \int_0^1 \mathbb{E}\left[q\partial_\mu G\left((q\mathbb{P})_{X^\theta}, X^\theta\right)\cdot(X'-X)\right]\mathrm{d}\theta,
\end{aligned}
\tag{57}
$$

with the convenient notation $X^\theta := \theta X' + (1-\theta)X$ and $q^\theta := \theta q' + (1-\theta)q$. Let us first prove the first line in A7. For a constant $C \geq 0$ and for $q$, $X$ and $X'$ satisfying $\mathbb{E}[q|X|^{2-r}], \mathbb{E}[q|X'|^{2-r}] \leq C$, we rewrite the second line in (57) as

$$
\begin{aligned}
G\left((q\mathbb{P})_{X'}\right) &= G\left((q\mathbb{P})_X\right) + \mathbb{E}\left[q\partial_\mu G\left((q\mathbb{P})_X, X\right)\cdot(X'-X)\right] \\
&\quad + \int_0^1 \mathbb{E}\left[q\left(\partial_\mu G\left((q\mathbb{P})_{X^\theta}, X^\theta\right) - \partial_\mu G\left((q\mathbb{P})_X, X\right)\right)\cdot(X'-X)\right]\mathrm{d}\theta.
\end{aligned}
\tag{58}
$$

By (46) we have

$$
\begin{aligned}
&\left|\partial_\mu G\left((q\mathbb{P})_{X^\theta}, X^\theta\right) - \partial_\mu G\left((q\mathbb{P})_X, X\right)\right| \\
&\leq L_C\left(1+|X|^{1-r}\right)\mathbb{E}\left[q|X-X'|^{2-r}\right]^{1/(2-r)} + L_C|X'-X|,
\end{aligned}
$$

from which we deduce, by Cauchy-Schwarz inequality, that

$$
\begin{aligned}
&\left|\int_0^1 \mathbb{E}\left[q\left(\partial_\mu G\left((q\mathbb{P})_{X^\theta}, X^\theta\right) - \partial_\mu G\left((q\mathbb{P})_X, X\right)\right)\cdot(X'-X)\right]\mathrm{d}\theta\right| \\
&\leq L'_C \mathbb{E}\left[q|X'-X|^2\right],
\end{aligned}
$$

for a constant $L'_C$ depending on $L_C$ and $C$. Inserting the above display in (58), this proves the first line in A7. We now establish the second line in A7. where the modulus of continuity $\varpi_L$ might increase. We rewrite the first line in (57) as

$$
\begin{aligned}
G\left((q'\mathbb{P})_X\right) &= G\left((q\mathbb{P})_X\right) + \int_{\mathbb{R}^n} \frac{\delta G}{\delta\mu}\left((q\mathbb{P})_X, x\right)\mathrm{d}\left[(q'\mathbb{P})_X - (q\mathbb{P})_X\right](x) \\
&+ \int_0^1 \left[\int_{\mathbb{R}^n}\left(\frac{\delta G}{\delta\mu}\left((q^\theta\mathbb{P})_X, x\right) - \frac{\delta G}{\delta\mu}\left((q\mathbb{P})_X, x\right)\right)\mathrm{d}\left[(q'\mathbb{P})_X - (q\mathbb{P})_X\right](x)\right]\mathrm{d}\theta.
\end{aligned}
\tag{59}
$$

By assumption (45),

$$
\begin{aligned}
\left|\frac{\delta G}{\delta\mu}\left((q^\theta\mathbb{P})_X, x\right) - \frac{\delta G}{\delta\mu}\left((q\mathbb{P})_X, x\right)\right| &\leq \left(1+|x|^{2-r}\right)\varpi\left(d_{2-r}\left((q^\theta\mathbb{P})_X, (q\mathbb{P})_X\right)\right) \\
&\leq \left(1+|x|^{2-r}\right)\varpi\left(\sup_\varphi \mathbb{E}\left[\varphi(X)(q-q')\right]\right) \\
&\leq \left(1+|x|^{2-r}\right)\varpi\left(\mathbb{E}\left[(1+|X|^{2-r})|q-q'|\right]\right),
\end{aligned}
$$

where $\varpi$ in the first line is the modulus of continuity of $\delta G/\delta\mu$ in the first argument, and is (here) independent of $x$ but depends on $q$, $q'$ and $X$ via $C$. As for $\varphi$ on the second line, it satisfies $|\varphi(x)| \leq 1 + |x|^{2-r}$.

Combining the last two displays, we obtain

$$G\left((q'\mathbb{P})_X\right) = G\left((q\mathbb{P})_X\right) + \int_{\mathbb{R}^n} \frac{\delta G}{\delta \mu}\left((q\mathbb{P})_X, x\right) \mathrm{d}\left[(q'\mathbb{P})_X - (q\mathbb{P})_X\right](x)$$
$$+ o\left(\mathbb{E}\left[(1+|X|^{2-r})|q-q'|\right]\right),$$

where $o(r)/|r| \to 0$ as $r$ tends to 0, uniformly in $q$, $q'$ and $X$ satisfying the two bounds $\mathbb{E}[q|X|^{2-r}], \mathbb{E}[q'|X|^{2-r}] \leq C$. This proves the second line in A7.

*Step 2: $\mathcal{G}$ verifies A8.* Consider again $(q,X),(q',X') \in \mathscr{G}$ such that $\mathbb{E}[q|X'|^{2-r}]$ and $\mathbb{E}[q'|X|^{2-r}] < +\infty$. By (58) and then (48), we have

$$G\left((q\mathbb{P})_{X'}\right) = G\left((q\mathbb{P})_X\right) + \mathbb{E}\left[q\partial_\mu G\left((q\mathbb{P})_X, X\right)\cdot(X'-X)\right]$$
$$+ \int_0^1 \frac{1}{\theta}\mathbb{E}\left[q\left(\partial_\mu G\left((q^\theta\mathbb{P})_X, X^\theta\right) - \partial_\mu G\left((q\mathbb{P})_X, X\right)\right)\cdot(X^\theta - X)\right]\mathrm{d}\theta$$
$$\geq G((q\mathbb{P})_X) + \mathbb{E}\left[q\partial_\mu G\left((q\mathbb{P})_X, X\right)\cdot(X'-X)\right],$$

which proves the second condition in A8. In order to establish the first condition in A8, we notice that, in (59),

$$(q^\theta\mathbb{P})_X = \theta(q'\mathbb{P})_X + (1-\theta)(q\mathbb{P})_X = (q\mathbb{P})_X + \theta\left((q'\mathbb{P})_X - (q\mathbb{P})_X\right),$$

and then,

$$G\left((q'\mathbb{P})_X\right) = G\left((q\mathbb{P})_X\right) + \int_{\mathbb{R}^n} \frac{\delta G}{\delta \mu}\left((q\mathbb{P})_X, x\right)\mathrm{d}\left[(q'\mathbb{P})_X - (q\mathbb{P})_X\right](x)$$
$$+ \int_0^1 \int_{\mathbb{R}^n} \frac{1}{\theta}\left[\frac{\delta G}{\delta \mu}\left((q^\theta\mathbb{P})_X, x\right) - \frac{\delta G}{\delta \mu}\left((q\mathbb{P})_X, x\right)\right]\mathrm{d}\left[(q^\theta\mathbb{P})_X - (q\mathbb{P})_X\right](x)\mathrm{d}\theta$$
$$\leq G((q\mathbb{P})_X) + \int_{\mathbb{R}^n} \frac{\delta G}{\delta \mu}\left((q\mathbb{P})_X, x\right)\mathrm{d}\left[(q'\mathbb{P})_X - (q\mathbb{P})_X\right](x),$$

where the last line follows by the monotonicity assumption (47). □

**Perspectives** *Common noise.* Our approach, based on the stochastic maximum principle, would allow us to introduce a common noise into the model in a direct manner. Similar to [31], we can think of an additive white noise manifesting in the form of an extra term $\sigma^0 \mathrm{d}W^0$ in the dynamics of $X$, where $W^0$ is a Brownian motion independent of $(W,\eta)$. Alternatively, we could randomize the coefficients independently of $(W,\eta)$. In any case, this additional source of randomness could be represented by tensorizing the space $\Omega$ (which carries the idiosyncratic noises) with a new space $\Omega^0$ (which carries the common noise). This approach is used in [36].

To incorporate this, the following changes would be necessary:

- The terminal cost would read

$$\mathbb{E}^0[\mathcal{G}(q_T(\omega^0,\cdot), X_T(\omega^0,\cdot))] = \mathbb{E}^0[G((q_T(\omega^0,\cdot)\mathbb{P})_{X_T(\omega^0,\cdot)})],$$

  for any element $\omega^0 \in \Omega^0$. This accounts for the fact that the common noise induces a conditioning.

- Assuming, without significant loss of generality, that the filtration on $\Omega^0$ is generated by a Brownian motion (denoted $W^0$), all the backward equations would include an additional penalization term in the form of a stochastic integral with respect to $W^0$, i.e., $\int_0^{\cdot} Z_s^0 \cdot \mathrm{d}W_s^0$. If the filtration were not Brownian, the penalization could instead be written as a (possibly discontinuous) martingale, which would make the model more complex to study.

- If the model were extended to incorporate risk aversion with respect to the common noise, the dynamics of $q^0$ would include an additional term of the form $q_s^0 Z_s^{0,\star} \cdot \mathrm{d}W_s^0$. As a consequence, the driver of the BSDE for $Y$ would also depend on the additional variable $Z^0$, where $Z^0$ arises from the martingale representation above. Accordingly, both the adjoint process and the Hamiltonian would have to be modified to account for this additional dependence.

*$N$-particles system.* A natural question is how the robust mean field model arises as the limit of an $N$-particle control problem. A thorough and rigorous analysis of this convergence process is beyond the scope of the present article and is left for future work. Nevertheless, we hope that the formal arguments provided in the second example of Subsection 3.2, as well as in the forthcoming examples presented in Subsection 4.2, will help the reader to identify, at least at an intuitive level, the underlying mechanisms from which the mean field model can be expected to emerge.

### 4.2 Examples

In this paragraph, we provide two examples that lead to a robust mean field control problem.

**Feynman-Kac path particle models.** Inspired by the monograph [48], we consider a large system of $N$ weakly interacting $d$-dimensional particles, with Gibbs distributions on the path space $\mathcal{C}([0,T],\mathbb{R}^n)^N$:

$$\exp\Bigg(\beta\Bigg[\frac{1}{2N}\sum_{i,j=1}^N G\left(X_T^{i,\psi^i}, X_T^{j,\psi^j}\right) + \frac{1}{2N}\sum_{i,j=1}^N \int_0^T F\left(X_t^{i,\psi^i}, X_t^{j,\psi^j}\right)\mathrm{d}t + \frac{1}{2}\sum_{i=1}^N\int_0^T |\psi_t^i|^2\mathrm{d}t\Bigg]\Bigg)\cdot \mathbb{P}^{\times N},$$

where $\beta > 0$ and $\mathbb{P}$ is the Wiener measure on $\Omega := \mathcal{C}([0,T],\mathbb{R}^n)$, $\psi^i : \mathcal{C}([0,T],\mathbb{R}^n)^N \to \mathbb{R}^d$ is a progressively-measurable control for each $i \in \{1,\dots,N\}$ and

$$X_t^{i,\psi^i}(\omega^1,\dots,\omega^N) = \omega_t^i + \int_0^t \psi_s^i(\omega^1,\dots,\omega^N)\mathrm{d}s, \quad t\in[0,T].$$

The goal is then to minimize, with respect to $(\psi^1,\dots,\psi^N)$, the free energy given (up to a logarithmic transformation) by

$$\mathbb{E}^{\times N}\Bigg[\exp\Bigg(\beta\Bigg[\frac{1}{2N}\sum_{i,j=1}^N G\left(X_T^{i,\psi^i}, X_T^{j,\psi^j}\right) + \frac{1}{2N}\sum_{i,j=1}^N \int_0^T F\left(X_t^{i,\psi^i}, X_t^{j,\psi^j}\right)\mathrm{d}t + \frac{1}{2}\sum_{i=1}^N\int_0^T |\psi_t^i|^2\mathrm{d}t\Bigg]\Bigg)\Bigg].$$

In order to simplify, we assume below that the running cost $F$ is equal to 0, but the analysis would be the same if $F$ were not trivial.

Thanks to Donsker-Varadhan's formula (see (15)) for a remainder, the free energy can be rewritten in the form

$$\begin{aligned}\sup_{q^{(N)}}\Bigg\{&\beta\mathbb{E}^{\times N}\left[q_T^{(N)}\left(\frac{1}{2N}\sum_{i,j=1}^N G\left(X_T^{i,\psi^i},X_T^{j,\psi^j}\right)+\frac{1}{2}\sum_{i=1}^N\int_0^T|\psi_t^i|^2\mathrm{d}t\right)\right]\\&-\mathrm{H}\left(q^{(N)}\mathbb{P}^{\times N}|\mathbb{P}^{\times N}\right)\Bigg\},\end{aligned}\tag{60}$$

where $q^{(N)}$ is taken in the space of densities on $\Omega^{\times N}$ with a finite entropy. We observe that the normalization in the potential is consistent with that used in the paragraph on risk measures in Subsection 3.2. This therefore constitutes a nonlinear version (in the sense that the potential now depends on the empirical measure through a second-order functional) of the previous example; for simplicity, this example is also presented in the case of uncontrolled volatility.

*Characterization of the saddle-point.* If $G$, viewed as a real-valued function on $\mathbb{R}^n\times\mathbb{R}^n$, is smooth, convex and at most of quadratic growth, then Theorem 10 applies to the minimization of the above quantity. The saddle point of the min-max problem (over $q^{(N)}$ and $(\psi^1,\ldots,\psi^N)$) can be characterized via a 6-tuple

$$\big(Y^{(N)},Z^{(N)},X^{(N)},p^{(N)},k^{(N)},q^{(N)}\big),$$

with

$$\begin{aligned}Z^{(N)}&=(Z^{(N),i})_{i=1,\ldots,N}, & X^{(N)}&=(X^{(N),i})_{i=1,\ldots,N},\\ p^{(N)}&=(p^{(N),i})_{i=1,\ldots,N}, & k^{(N)}&=(k^{(N),i,j})_{i,j=1,\ldots,N},\end{aligned}$$

solution of (using the notation $B_t^j(\omega^1,\ldots,w^N):=\omega_t^j$, for $j=1,\ldots,N$ and $t\in[0,T]$)

$$\left\{\begin{aligned}-\mathrm{d}Y_t^{(N)}&=\left(\tfrac{1}{2}\textstyle\sum_{j=1}^N|Z^{(N),j}|^2+\tfrac{\beta}{2}\sum_{j=1}^N|\psi_t^j|^2\right)\mathrm{d}t-\textstyle\sum_{j=1}^N Z_t^{(N),j}\cdot\mathrm{d}B_t^j,\\ \mathrm{d}X_t^{(N),i}&=\psi_t^i\mathrm{d}t+\mathrm{d}B_t^i,\\ -\mathrm{d}p_t^{(N),i}&=-\textstyle\sum_{i=1}^N k_t^{(N),i,j}\cdot\mathrm{d}B_t^j,\\ \mathrm{d}q_t^{(N)}&=q_t^{(N)}\textstyle\sum_{i=1}^N Z_t^{(N),i}\cdot\mathrm{d}B_t^i,\end{aligned}\right.\tag{61}$$

for $t\in[0,T]$, with the optimality condition $\psi_t^i=-[\beta q_t^{(N)}]^{-1}p_t^{(N),i}$ and the boundary conditions

$$\left\{\begin{aligned}Y_T^{(N)}&=\tfrac{\beta}{2N}\textstyle\sum_{i,j=1}^N G\left(X_T^{(N),i},X_T^{(N),j}\right)\\ p_T^{(N),i}&=q_T^{(N)}\tfrac{\beta}{2N}\textstyle\sum_{j=1}^N\left[\partial_x G\left(X_T^{(N),i},X_T^{(N),j}\right)+\partial_y G\left(X_T^{(N),j},X_T^{(N),i}\right)\right].\end{aligned}\right.\tag{62}$$

Similar to the discussion initiated in Subsection 3.2, the question here is to understand, at least informally, how the above system is connected to the mean field control problem described in Subsection 4.1. To better appreciate the intuitive arguments that we present, it is worth mentioning from the analysis carried out in Subsection 5.2 (see in particular Lemma 27) that the solution of the BSDE (24) is understood via the product $qY=(q_tY_t)_{t\in[0,T]}$. This prompts us to consider, here, the product

$q^{(N)}Y^{(N)}$. The aforementioned Lemma 27 says that $q^{(N)}Y^{(N)}$ is a semi-martingale under the probability measure $q_T^{(N)}\mathbb{P}$, satisfying

$$\left\{\begin{array}{ll} -\mathrm{d}\left[q_t^{(N)}Y_t^{(N)}\right] & = \left(-\frac{1}{2}\sum_{j=1}^N q_t^{(N)}|Z^{(N),j}|^2 + \frac{\beta}{2}\sum_{j=1}^N q_t^{(N)}|\psi_t^j|^2\right)\mathrm{d}t \\ & \quad -q_t^{(N)}(1+Y_t^{(N)})\sum_{j=1}^N Z_t^{(N),j}\cdot \mathrm{d}B_t^j, \\ q_T^{(N)}Y_T^{(N)} & = \frac{\beta}{2N}q_T^N\sum_{i,j=1}^N G\left(X_T^{(N),i}, X_T^{(N),j}\right). \end{array}\right. \tag{63}$$

*Corresponding robust MFC problem.* In parallel, consider the robust mean field control problem (MFC), which we recall below for convenience:

$$\inf_{\psi}\sup_{q}\left\{\mathbb{E}\left[\frac{\beta}{2}\int_{\mathbb{R}^n} G\left(x,y\right)\left(q_T\mathbb{P}_{X_T^\psi}\right)^{\otimes 2}(\mathrm{d}x,\mathrm{d}y) + \frac{\beta}{2}\int_0^T q_s|\psi_s|^2\mathrm{d}s\right] - H\left(q_T\mathbb{P}|\mathbb{P}\right)\right\},$$

where for simplicity we do not specify the sets of admissibility in which $q$ and $\psi$ are taken. Generally speaking, this problem is defined on the original probability space $(\Omega,\mathcal{F},\mathbb{P})$ equipped with the Brownian motion $W$, but we can consider, for each $i=1,\dots,N$, the same problem but on the $i$th factor $\Omega^N$ and thus with respect to the Brownian motion $B^i$ instead of $W$. For each $i=1,\dots,N$, we then call $(\tilde{q}^i,\tilde{X}^i,\tilde{\psi}^i)$ the saddle point of the corresponding problem, which exists and is unique under the assumptions of Corollary 15. At this stage, these assumptions are taken for granted, but we will discuss its meaning in more depth at the end of this paragraph. Importantly, we observe that $(\tilde{q}^i,\tilde{X}^i,\tilde{\psi}^i)$ is a function of the sole $\omega^i$. The Pontryagin system characterizing $(\tilde{q}^i,\tilde{X}^i,\tilde{\psi}^i)$ reads

$$\left\{\begin{array}{ll} -\mathrm{d}\tilde{Y}_t^i & = \left(\frac{1}{2}|\tilde{Z}_t^i|^2 + \frac{\beta}{2}|\tilde{\psi}_t^i|^2\right)\mathrm{d}t - \tilde{Z}_t^i\cdot\mathrm{d}B_t^i, \\ \mathrm{d}\tilde{X}_t^i & = \tilde{\psi}_t^i\mathrm{d}t + \mathrm{d}B_t^i, \\ -\mathrm{d}\tilde{p}_t^i & = -\tilde{k}_t^i\cdot\mathrm{d}B_t^i, \\ \mathrm{d}\tilde{q}_t^i & = \tilde{q}_t^i\tilde{Z}_t^i\cdot\mathrm{d}B_t^i, \end{array}\right. \tag{64}$$

for $t\in[0,T]$, with the optimality condition $\tilde{\psi}_t^i = -[\beta\tilde{q}_t^i]^{-1}\tilde{p}_t^i$ and the boundary conditions

$$\left\{\begin{array}{ll} \tilde{Y}_T^i & = \frac{\beta}{2}\int_{\mathbb{R}^n}\left[G(\tilde{X}_T^i,z)+G(z,\tilde{X}_T^i)\right]\mathrm{d}(\tilde{q}_T^i\mathbb{P})_{\tilde{X}_T^i}(z) \\ \tilde{p}_T^i & = \frac{\beta}{2}\tilde{q}_T^i\int_{\mathbb{R}^n}\left[\partial_x G\left(\tilde{X}_T^i,z\right)+\partial_y G\left(z,\tilde{X}_T^i\right)\right]\mathrm{d}(\tilde{q}_T^i\mathbb{P})_{\tilde{X}_T^i}(z). \end{array}\right. \tag{65}$$

We then let

$$\tilde{Y}_t^{(N)} = \sum_{i=1}^N \tilde{Y}_t^i, \quad \tilde{q}_t^{(N)} = \prod_{i=1}^N q_t^i, \quad t\in[0,T],$$

and, following (63), we consider the process $\tilde{q}^{(N)}\tilde{Y}^{(N)}$. It satisfies

$$\left\{\begin{array}{ll} -\mathrm{d}\left[\tilde{q}_t^{(N)}\tilde{Y}_t^{(N)}\right] & = \left(-\frac{1}{2}\tilde{q}_t^{(N)}\sum_{i=1}^N|\tilde{Z}^i|^2 + \frac{\beta}{2}\tilde{q}_t^{(N)}\sum_{i=1}^N|\tilde{\psi}_t^i|^2\right)\mathrm{d}t \\ & \quad -\tilde{q}_t^{(N)}(1+Y_t^{(N)})\sum_{i=1}^N\tilde{Z}_t^i\cdot\mathrm{d}B_t^i, \\ \tilde{q}_T^{(N)}\tilde{Y}_T^{(N)} & = \frac{\beta}{2}\tilde{q}_T^{(N)}\sum_{i=1}^N\int_{\mathbb{R}^n}\left[G(\tilde{X}_T^i,z)+G(z,\tilde{X}_T^i)\right]\mathrm{d}(\tilde{q}_T^i\mathbb{P})_{\tilde{X}_T^i}(z). \end{array}\right. \tag{66}$$

*Connecting the two problems.* Of course, in the above right-hand side, $(\tilde{q}_T^i \mathbb{P})_{\tilde{X}_T^i}$ is independent of $i$ and can be replaced by $(\tilde{q}_T^1 \mathbb{P})_{\tilde{X}_T^1}$. For simplicity, we remove below the index 1 and merely write $(\tilde{q}_T \mathbb{P})_{\tilde{X}_T}$. The connection between the form of the boundary condition for $\tilde{q}_T^{(N)} \tilde{Y}_T^{(N)}$ in (66) and the form of the boundary condition for $q_T^{(N)} Y_T^{(N)}$ in (63) can be better understood by applying the weak law of large numbers under the probability $\tilde{q}_T^{(N)} \mathbb{P}$. Indeed, since the random variables $\tilde{X}_T^1, \dots, \tilde{X}_T^N$ are independent under $\tilde{q}_T^{(N)}$, with $(\tilde{q}\mathbb{P})_{\tilde{X}_T}$ as common distribution, and because $\int_{\mathbb{R}^n \times \mathbb{R}^n} G(x,y) (\tilde{q}_T \mathbb{P}_{\tilde{X}_T})^{\otimes 2}(\mathrm{d}x, \mathrm{d}y) < +\infty$ (as a consequence of Lemma 40), we have

$$\lim_{N \to +\infty} \mathbb{E}\Bigg[ \tilde{q}_T^{(N)} \Bigg| \frac{1}{N^2} \sum_{i,j=1}^N G(\tilde{X}_T^i, \tilde{X}_T^j) - \frac{1}{2N} \sum_{i=1}^N \int_{\mathbb{R}^n} \Big( G(\tilde{X}_T^i, z) + G(z, \tilde{X}_T^i) \Big) (\tilde{q}_T \mathbb{P})_{\tilde{X}_T})(\mathrm{d}z) \Bigg| \Bigg] = 0. \tag{67}$$

Pay attention to the fact that the boundary conditions for $\tilde{q}_T^{(N)} \tilde{Y}_T^{(N)}$ in (66) and $q_T^{(N)} Y_T^{(N)}$ in (63) are of order $N$, whereas the two terms in the above difference are of order 1. That said, the above display shows that $\tilde{q}_T^{(N)} \tilde{Y}_T^{(N)}$ satisfies a boundary condition similar to the one satisfied by $q_T^{(N)} Y_T^{(N)}$ in (63), up to a remainder of order $o(N) = \varepsilon_N N$ with $\varepsilon_N$ converging to 0 in $L^1$ under $\mathbb{P}$. This makes it possible to view the process $\tilde{q}^{(N)} \tilde{Y}^{(N)}$ as a 'nearly solution' of the equation satisfied by $q^{(N)} Y^{(N)}$, but with $\psi^i$ replaced by $\tilde{\psi}^i$ in the generator of the backward component, and $(X_T^{(N),1}, \dots, X_T^{(N),N})$ replaced by $(\tilde{X}_T^1, \dots, \tilde{X}_T^N)$ in the terminal condition.

By the same argument, one can multiply each $\tilde{p}^i$ in (64), for $i \in \{1, \dots, N\}$, by $\tilde{q}^{(N)} (\tilde{q}^i)^{-1}$. The resulting process $\tilde{q}^{(N)} (\tilde{q}^i)^{-1} \tilde{p}^i$ remains a local martingale, and its boundary condition satisfies, up to a new remainder of order $o(N)$, a boundary condition similar to the one satisfied by $p^{(N),i}$ in (62). This shows that the process $\tilde{q}^{(N)} (\tilde{q}^i)^{-1} \tilde{p}^i$ is a 'nearly solution' of the equation satisfied by $p^{(N),i}$, but with $(X_T^{(N),1}, \dots, X_T^{(N),N})$ replaced by $(\tilde{X}_T^1, \dots, \tilde{X}_T^N)$. Next, rewriting the identity

$$\tilde{\psi}_t^i = -(\beta \tilde{q}_t^i)^{-1} \tilde{p}_t^i$$

in the form

$$\tilde{\psi}_t^i = -(\beta \tilde{q}_t^{(N)})^{-1} \tilde{q}_t^{(N)} (\tilde{q}_t^i)^{-1} \tilde{p}_t^i,$$

we observe that $\tilde{\psi}^i$ can be expressed in terms of $\tilde{q}^{(N)}$ and $\tilde{q}^{(N)} (\tilde{q}^i)^{-1} \tilde{p}^i$ via the same function that allows one to express $\psi^i$ in terms of $q^{(N)}$ and $p^{(N),i}$.

Altogether, this shows that the tuple

$$(\tilde{Y}^{(N)}, \tilde{Z}^1, \dots, \tilde{Z}^N, \tilde{X}^1, \dots, \tilde{X}^N, \tilde{q}^{(N)}, \tilde{q}^{(N)} (\tilde{q}^1)^{-1} \tilde{p}^1, \dots, \tilde{q}^{(N)} (\tilde{q}^N)^{-1} \tilde{p}^N)$$

is a nearly solution of the forward–backward system solved by the tuple

$$(Y^{(N)}, Z^{(N),1}, \dots, Z^{(N),N}, X^{(N),1}, \dots, X^{(N),N}, p^{(N),1}, \dots, p^{(N),N}, q^{(N)}),$$

which makes the connection between (60) and the robust MFC problem.

*Assumptions on $G$.* We now comment on the assumptions needed to apply Corollary 15 in the analysis of the robust MFC problem.

Quite surprisingly, although the convexity of $G$ suffices to apply Theorem 10 in order to characterize the saddle points of (60) –because the map $(q, x_1, \dots, x_n) \mapsto q \sum_{i,j=1} G(x_i, x_j)$ is linear in $q$ and convex in $(x_1, \dots, x_n)$–, it does not suffice to apply Corollary 15. Indeed, the map

$$\mu \in \mathcal{P}_2(\mathbb{R}^n) \mapsto \int_{(\mathbb{R}^n)^2} G(x,y) \, \mathrm{d}\mu^{\times 2}(x,y)$$

is displacement convex –as a consequence of the convexity of $G$– but may fail to be flat concave. For instance, if $G(x,y) = \varphi(x)\varphi(y)$, for some non-negative convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$, then $G$ is convex. However, for any $\mu \in \mathcal{P}_2(\mathbb{R}^n)$,

$$\Gamma(\mu) := \int_{(\mathbb{R}^n)^2} G(x,y) \, \mathrm{d}\mu^{\times 2}(x,y) = \left( \int_{\mathbb{R}^n} \varphi(x) \, \mathrm{d}\mu(x) \right)^2,$$

which shows that the function $\Gamma : \mu \in \mathcal{P}_2(\mathbb{R}^n) \mapsto \Gamma(\mu)$ is flat convex.

Additional conditions are therefore required to apply Corollary 15. Although, in the previous paragraph, we did not provide a complete proof but only some intuition to justify the passage from (61)–(62) to (64)–(65), we believe that the need for extra assumptions to ensure existence and uniqueness of a solution to the mean field problem reflects the price to pay for passing to the limit (as $N \to \infty$) in the original problem (60).

Following the discussion in the previous subsection, we now provide an example of a class of convex functions $G$ for which $\mu \in \mathcal{P}_2(\mathbb{R}^n) \mapsto \Gamma(\mu)$ is flat concave. If $G$ itself is not convex but $\Gamma$ is flat concave, one may replace $G$ by the function $(x,y) \mapsto G(x,y) + a(|x|^2 + |y|^2)$, for $a > 0$ large enough, in order to enforce displacement convexity while preserving flat concavity. Thus, the remaining task is to provide an example of a function $G$ for which $\Gamma$ is concave. One such example is given by any function of the form

$$(x,y) \mapsto -\sum_{i=1}^{k} \lambda_i \, h_i(x) \, h_i(y),$$

where $k \geq 0$, $\lambda_i > 0$, and $h_i$ is a smooth function with bounded derivative, for each $i = 1, \dots, k$.

**Robust approximation of a Gibbs measure on the path space.** We now present another example, building on the previous one, but which corresponds to the robustification, with respect to the central planner's strategy $\psi$, of a control problem defined on Nature's state $q$. It is inspired by recent works on stochastic algorithms (a more precise list of references is given below).

Given a potential $\mathcal{W}$ defined on the Wiener path space $\Omega := \mathcal{C}([0,T], \mathbb{R}^d)$, one wants to approximate the normalized Gibbs probability measure

$$\mathbb{P}^{\text{target}} := \frac{1}{\mathcal{Z}} \exp\left(-\mathcal{W}\right) \mathbb{P}, \qquad \text{with} \quad \mathcal{Z} := \mathbb{E}[\exp(-\mathcal{W})], \tag{68}$$

by the law of a controlled diffusion process of the form (say to simplify that $X_0^\phi = 0$)

$$\mathrm{d}X_t^\phi = \phi_t \mathrm{d}t + \mathrm{d}B_t, \quad t \in [0,T]. \tag{69}$$

(Here, we use the notation $B$ instead of $W$ for the canonical process, with is a Brownian motion under the Wiener measure $\mathbb{P}$; this to avoid confusion with the potential $\mathcal{W}$.) Typically, $\mathcal{W}(\omega)$, where $\omega = (\omega_t)_{t\in[0,T]}$ denotes the generic element of the space $\Omega$, is chosen as

$$\mathcal{W}(\omega) = G(\omega_\tau) + \int_0^\tau F(\omega_t)\mathrm{d}t, \tag{70}$$

where $\tau$ is the realization, at $\omega$, of a stopping time, usually chosen as the first exit time of $\omega$ from a given domain. Obviously, the structure of $\mathcal{W}$ described above is especially adapted to Markovian dynamics, which leads us to choose, in this situation, the control $(\phi_t)_{t\in[0,T]}$ in a Markov feedback form $(\phi_t = \Phi(t, X_t))_{t\in[0,T]}$.

*Exact solution to the targeting problem.* In fact, under standard assumptions covering the Markovian framework, one can find a control $\bar{\psi}$ such that the law $\mathbb{P}\circ(X^{\bar{\psi}})^{-1}$ of $(X_t^{\bar{\psi}})_{t\in[0,T]}$ under $\mathbb{P}$ perfectly matches the target distribution $\mathbb{P}^{\text{target}}$, i.e.

$$\mathbb{P}\circ(X^{\bar{\psi}})^{-1} = \mathbb{P}^{\text{target}}. \tag{71}$$

Assume indeed that one can solve the FBSDE system (for simplicity, we do not specify the spaces in which solutions are taken because this would be useless for the rest of the paragraph)

$$\begin{cases} -\mathrm{d}Y_t &= \frac{1}{2}|Z_t|^2\mathrm{d}t - Z_t\cdot\mathrm{d}B_t, \quad Y_T = \mathcal{W}(X), \\ \mathrm{d}X_t &= -Z_t\mathrm{d}t + \mathrm{d}B_t, \qquad\quad X_0 = 0. \end{cases} \tag{72}$$

Then, the backward equation can be reformulated as

$$\exp\left(-\mathcal{W}(X)\right)\mathcal{E}_T\left(\int_0^\cdot Z_r\cdot\mathrm{d}B_r\right) = \exp\left(-Y_0\right), \tag{73}$$

where we recall that $Y_0$ is deterministic (as it is the initial value of the BSDE in (71)). We deduce that, for any bounded and measurable function $\Phi : \mathcal{C}([0,T],\mathbb{R}^d)\to\mathbb{R}$,

$$\mathbb{E}\left[\exp\left(Y_0 - \mathcal{W}(X)\right)\mathcal{E}_T\left(\int_0^\cdot Z_r\cdot\mathrm{d}B_r\right)\Phi(X)\right] = \mathbb{E}\left[\Phi(X)\right].$$

Thanks to the forward equation in (72) and provided that the Girsanov transformation can be rigorously applied, we observe that the left-hand side is equal to $\mathbb{E}[e^{Y_0-\mathcal{W}}\Phi]$, because the law of $X$ under $\mathcal{E}_T(\int_0^\cdot Z_r\cdot\mathrm{d}B_r)\mathbb{P}$ is the same as the law of $B$ under $\mathbb{P}$. Since the function $\Phi$ is arbitrary, this proves that the law of $X$ under $\mathbb{P}$ is the Gibbs measure $\mathbb{P}^{\text{target}}$, as required.

The analysis of the FBSDE (72) is standard in the Markovian setting. In this case, there exists a function $\Psi$, given as the solution of an auxiliary nonlinear parabolic PDE (see [35, Chapter 3]), such that $(Z_t = -\Psi(t, X_t))_{t\in[0,T]}$. In particular, one can express $\mathbb{P}\circ X^{-1}$ as $\mathcal{E}_T(\int_0^\cdot\psi_t\cdot\mathrm{d}B_t)\mathbb{P}$, where $\psi_t(\omega) = \Psi(t,\omega_t)$ (the latter is different from $-Z_t(\omega) = \Psi(t, X_t(\omega))$).

*Reformulation as a Nature optimization problem.* Interestingly, this targeting problem can be recast as a minimization problem in the space of probability measures. Indeed, using Donsker-Varadhan's lemma, it holds, for any control $(\phi_t)_{t\in[0,T]}$ such that the measure $\mathbb{P}^\phi := \mathcal{E}_T(\int_0^\cdot\phi_t\cdot\mathrm{d}B_t)\mathbb{P}$ has a relative finite entropy $H(\mathbb{P}^\phi|\mathbb{P})$,

$$-\ln\left(\mathbb{E}\left[\exp(-\mathcal{W})\right]\right) \le \mathbb{E}\left[\frac{\mathrm{d}\mathbb{P}^\phi}{\mathrm{d}\mathbb{P}}\mathcal{W}\right] + \mathrm{H}\left(\mathbb{P}^\phi\middle|\mathbb{P}\right). \tag{74}$$

When $\phi$ is equal to $\psi$, the right-hand side becomes

$$\mathbb{E}\left[\frac{\mathrm{d}\mathbb{P}^\psi}{\mathrm{d}\mathbb{P}}\mathcal{W}\right]+\mathrm{H}\left(\mathbb{P}^\psi\middle|\mathbb{P}\right)=\mathbb{E}\left[\mathcal{W}\left(X^\psi\right)\right]+\mathrm{H}\left(\mathbb{P}\circ(X^\psi)^{-1}\middle|\mathbb{P}\right)$$
$$=\mathbb{E}\left[\mathcal{W}(X)+\frac{1}{2}\int_0^T|Z_t|^2\mathrm{d}t\right],$$

which is equal (thanks to (72)) to $\mathbb{E}[Y_0]=\mathbb{E}[\exp(-\mathcal{W})]$ (with the latter following from (73) and a new application of Girsanov's formula). Therefore, $\psi$ solves the minimization problem

$$\inf_\phi\left\{\mathbb{E}\left[\frac{\mathrm{d}\mathbb{P}^\phi}{\mathrm{d}\mathbb{P}}\mathcal{W}\right]+\mathrm{H}\left(\mathbb{P}^\phi\middle|\mathbb{P}\right)\right\}, \tag{75}$$

hence connecting the targeting problem (71) and the minimization problem (75). For example, these two problems are tackled in control based importance sampling methods for diffusion processes (see for instance [81, 90] and [92, Chapter 6], from which we borrowed part of the presentation) and in diffusion based models for generative adversarial networks (see for instance the fine tuning analysis provided in [94, 96] and the MFC interpretation of score matching approaches [98]).

In fact, the connection between the targeting problem (71) and the minimization problem (75) can be better understood by reformulating the latter, and then by observing that $\psi$ solves

$$\inf_\phi \mathrm{H}\left(\mathbb{P}^\phi\middle|\frac{1}{\mathcal{Z}}e^{-\mathcal{W}}\mathbb{P}\right), \tag{76}$$

the optimal value being equal to 0. Above, we recall that $\mathcal{Z}=\mathbb{E}[\exp(-W)]$. Rephrased in our framework, the density $\mathrm{d}\mathbb{P}^\phi/\mathrm{d}\mathbb{P}$ appearing in both (75) and (76) must be identified with Nature's state $q_T$ at terminal time. Therefore, the two problems can be regarded as optimal control problem for Nature (even though the original problem (71)) is formulated as a control problem for the player).

*Robust version.* Now, consistently with the robust approach introduced in this work, one can think of a situation where there is some uncertainty on the precise form of the potential $\mathcal{W}$ in the targeting measure $e^{-\mathcal{W}}\cdot\mathbb{P}$ in (68). We thus change $\mathcal{W}$ into $\mathcal{W}(X^\psi)$, with $X^\psi$ as in (69). Intuitively, this says that there is some uncertainty on the 'observed values' of $\mathcal{W}$, say for instance because the potential is computed along an approximation of the canonical process (as in the stochastic algorithms cited above).

Next, we introduce two related min-max problems. The first problem is a robust version of (75):

$$\inf_{q\in\mathcal{Q}}\sup_{\psi\in\mathcal{A}}\left\{\mathrm{H}\left(\mathbb{Q}^q|\,\mathbb{P}\right)+\mathbb{E}\left[q_T\mathcal{W}(X^\psi)-\frac{1}{2}q_T\int_0^T|\psi_t|^2\mathrm{d}t\right]\right\},\quad \mathbb{Q}^q=q_T\mathbb{P}, \tag{77}$$

which is quite similar to the second example in Subsection 3.2 and to the first example in this subsection. The second one is a robust version of (76):

$$\inf_{q\in\mathcal{Q}}\sup_{\psi\in\mathcal{A}}\left\{\mathrm{H}\left(\mathbb{Q}^q\middle|\mathbb{G}^\psi\right)-\frac{1}{2}\mathbb{E}\left[q_T\int_0^T|\psi_t|^2\mathrm{d}t\right]\right\},\quad \mathbb{G}^\psi=\frac{1}{\mathcal{Z}^\psi}e^{-\mathcal{W}(X^\psi)}\mathbb{P}, \tag{78}$$

with $\mathcal{Z}^\psi:=\mathbb{E}[\exp(-\mathcal{W}(X^\psi))]$.

The two problems are not the same because of the presence of the normalization constant in the second one. In both situations, the penalty term $-\frac{1}{2}q_T\int_0^T|\psi_t|^2\mathrm{d}t$ should be regarded as a regularization of the problem that just amounts in replacing the original potential $\mathcal{W}(X^\psi)$ by the effective one $\mathcal{W}(X^\psi)-\frac{1}{2}q_T\int_0^T|\psi_t|^2\mathrm{d}t$. Also, the term $\mathcal{W}(X^\psi)$ itself can be chosen as a function of the realization of the path but also of its statistical distribution under $\mathbb{P}$. For instance, a typical choice, consistent with the previous example on Feynman-Kac models, is

$$\mathcal{W}(X_T^\psi)=W\left(X_T^\psi,\mathbb{P}\circ(X_T^\psi)^{-1}\right).$$

*Explicit computation.* When $\mathcal{W}$ is (say) a concave function of the terminal state, the first problem (77) satisfies our concavity-convexity conditions and there exists a (unique) saddle point to the min-max problem. To better illustrate the result, we just focus on the case when $d=1$ and $\mathcal{W}(\omega)=-\beta\omega_T$, for some parameter $\beta\in\mathbb{R}$. Then, very similar to the risk averse portfolio management problem addressed in Subsection 3.2, the unique saddle point $(\bar{q},\bar{\psi})$ can be found explicitly. Here,

$$\bar{\psi}_t=-\beta, \tag{79}$$

and

$$\bar{q}_T=\frac{1}{\bar{\mathcal{Z}}}\exp\left(-\mathcal{W}(X^{\bar{\psi}})-\frac{1}{2}\int_0^T|\bar{\psi}_t|^2\mathrm{d}t\right)=\frac{1}{\bar{\mathcal{Z}}}\exp\left(\beta(B_T-T\beta)-\frac{1}{2}T\beta^2\right), \tag{80}$$

where

$$\bar{\mathcal{Z}}=\mathbb{E}\left[\exp\left(\beta(B_T-T\beta)-\frac{1}{2}T\beta^2\right)\right]=\exp(-T\beta^2). \tag{81}$$

The second problem (78) is more difficult to handle. Using the explicit form of $\mathcal{Z}^\psi$ (and expanding the various logarithms inside the definition of the entropy), it can be rewritten as

$$\inf_{q\in\mathcal{Q}}\sup_{\psi\in\mathcal{A}}\Bigg\{\mathbb{E}\left[q_T\mathcal{W}(X^\psi)-\frac{1}{2}q_T\int_0^T|\psi_t|^2\mathrm{d}t\right] + \ln\left(\mathbb{E}\left[\exp\left(-\mathcal{W}(X^\psi)\right)\right]\right)+\mathrm{H}\left(\mathbb{Q}^q|\mathbb{P}\right)\Bigg\}.$$

Here we recall from (74) that the cumulant generating function appearing on the second line of the right-hand can be reformulated as the supremum (over $\phi$) of $\mathbb{E}[-(\mathrm{d}\mathbb{P}^\phi/\mathrm{d}\mathbb{P})\mathcal{W}(X^\psi)]-\mathrm{H}(\mathbb{P}^\phi|\mathbb{P})$. In particular, if $\mathcal{W}$ is concave, then $-\mathcal{W}$ is convex (in $X_T^\psi$) and the supremum (over $\phi$) is also convex in $X_T^\psi$. As a result, it is not clear whether the cost is concave in $\psi$, which prevents any application of the results obtained in the article.

Nevertheless, one can use the saddle point $(\bar{q},\bar{\psi})$ obtained for the problem (77) in order to gain some insight into the problem (78). Indeed, by the saddle point property, we have

$$\mathbb{E}\left[\bar{q}_T\mathcal{W}\left(X^{\bar{\psi}}\right)-\frac{1}{2}\bar{q}_T\int_0^T|\bar{\psi}_t|^2\mathrm{d}t\right]=\sup_\psi\mathbb{E}\left[\bar{q}_T\mathcal{W}\left(X^\psi\right)-\frac{1}{2}\bar{q}_T\int_0^T|\psi_t|^2\mathrm{d}t\right].$$

By concavity of the cost function (with respect to $\psi$), one deduces that, for any $\psi$

$$\mathbb{E}\left[\bar{q}_T\mathcal{W}\left(X^{\bar{\psi}}\right) - \frac{1}{2}\bar{q}_T\int_0^T |\bar{\psi}_t|^2\mathrm{d}t\right] \geq \mathbb{E}\left[\bar{q}_T\mathcal{W}\left(X^{\psi}\right) - \frac{1}{2}\bar{q}_T\int_0^T |\psi_t|^2\mathrm{d}t\right] + \frac{1}{2}\mathbb{E}\left[\bar{q}_T\int_0^T |\psi_t - \bar{\psi}_t|^2\mathrm{d}t\right].$$

And then, by expanding the logarithm inside the definition of the entropy,

$$\begin{aligned} &\mathrm{H}\left(\mathbb{Q}^{\bar{q}}\middle|\mathbb{G}^{\bar{\psi}}\right) - \frac{1}{2}\mathbb{E}\left[\bar{q}_T\int_0^T |\bar{\psi}_t|^2\mathrm{d}t\right] \\ &= \mathbb{E}\left[\bar{q}_T\mathcal{W}\left(X^{\bar{\psi}}\right) - \frac{1}{2}\bar{q}_T\int_0^T |\bar{\psi}_t|^2\mathrm{d}t\right] + \ln\left(\mathbb{E}\left[\exp\left(-\mathcal{W}(X^{\bar{\psi}})\right)\right]\right) + \mathrm{H}\left(\mathbb{Q}^{\bar{q}}|\mathbb{P}\right) \\ &\geq \mathrm{H}\left(\mathbb{Q}^{\bar{q}}\middle|\mathbb{G}^{\psi}\right) - \frac{1}{2}\mathbb{E}\left[\bar{q}_T\int_0^T |\psi_t|^2\mathrm{d}t\right] + \Delta(\psi,\bar{\psi}), \end{aligned}$$

with

$$\Delta(\psi,\bar{\psi}) := -\ln\left(\frac{\mathbb{E}\left[\exp\left(-\mathcal{W}(X^{\psi})\right)\right]}{\mathbb{E}\left[\exp\left(-\mathcal{W}(X^{\bar{\psi}})\right)\right]}\right) + \frac{1}{2}\mathbb{E}\left[\bar{q}_T\int_0^T \left|\psi_t - \bar{\psi}_t\right|^2\mathrm{d}t\right],$$

which gives a way to control the variation $\Delta(\psi,\bar{\psi})$ of the cost when $\psi$ is deviating from $\bar{\psi}$. We can illustrate this idea in this example, by means of in (79)–(80)–(81). We have

$$\begin{aligned} \Delta(\psi,\bar{\psi}) &= -\ln\left(\frac{\mathbb{E}\left[\exp\left(\beta X^{\psi}\right)\right]}{\mathbb{E}\left[\exp\left(\beta X^{\bar{\psi}}\right)\right]}\right) + \frac{1}{2}\mathbb{E}\left[\bar{q}_T\int_0^T \left|\psi_t - \bar{\psi}_t\right|^2\mathrm{d}t\right] \\ &= -\ln\left\{\mathbb{E}\left[\exp\left(\beta B_T - \frac{1}{2}\beta^2 T\right)\exp\left(\beta\int_0^T \left[\psi_t - \bar{\psi}_t\right]\mathrm{d}t\right)\right]\right\} \\ &\quad + \frac{1}{2}\mathbb{E}\left[\exp\left(\beta B_T - \frac{1}{2}\beta^2 T\right)\int_0^T \left|\psi_t - \bar{\psi}_t\right|^2\mathrm{d}t\right]. \end{aligned}$$

This gives a way to control the output performance in terms of the disturbance, which principle is underpinning the theory of $H^\infty$-control (see [6]).

## 4.3 Variational mean field games

In this section, we formulate a mean field game problem that is closely related to the robust mean field control problem introduced in the previous section, and that even derives from it for some specific choice of the coefficients. The latter situation is an extension, to the robust setting, of the connection that exists between mean field control problems and potential mean field games.

Generally speaking, a mean field game is defined as a fixed point problem on the distribution of a control problem (with the latter being solved by a so-called representative agent in a continuum of agents). In our case, the fixed point problem is set on a generic non-negative measure $\mu \in \mathcal{M}_{2-r}(\mathbb{R}^n)$; given $\mu$, the representative agent minimizes a risk-averse objective functional

$$\inf_{\psi\in\mathcal{A}}\sup_{q\in\mathcal{Q}}\mathcal{J}[\mu](q,\psi), \quad \mathcal{J}[\mu](q,\psi) := \mathbb{E}\left[q_T g(\mu, X_T^{\psi}) + \int_0^T q_s \ell(s,\psi_s)\mathrm{d}s\right] - \mathcal{S}(q), \quad (\mathrm{P}_\mu)$$

where the controlled state process $(X_t^\psi)_{t\in[0,T]}$ satisfies the dynamics given in (7). Assuming that, for each $\mu \in \mathcal{M}^{2-r}(\mathbb{R}^n)$, the function $(q, X) \mapsto qg(\mu, X)$ satisfies the assumption of Theorem 10 (we clarify the choice of $g$ right below), we can denote by $\psi^\mu$ and $q^\mu$ the optimal controls of the representative agent and of Nature, respectively. The fixed point condition requires that the measure $\mu$ coincides with the law of the terminal state $X_T^{\psi^\mu}$ under the measure $q_T^\mu \mathbb{P}$ induced by Nature, that is,

$$\mu = \big(q_T^\mu \mathbb{P}\big) \circ (X_T^{\psi^\mu})^{-1}. \tag{MFG-eq}$$

The mean field game problem thus consists in finding a triple $(q, \psi, \mu) \in \mathcal{Q} \times \mathcal{A} \times \mathcal{M}_{2-r}(\mathbb{R}^n)$ such that

$$\mathcal{J}[\mu](q, \psi) = \inf_{\psi' \in \mathcal{A}} \sup_{q' \in \mathcal{Q}} \mathcal{J}[\mu](q', \psi'), \quad \mu = (q_T \mathbb{P}) \circ (X_T^\psi)^{-1}. \tag{MFG}$$

Here are the assumptions required on $g$.

A10 We assume that there exists a function $G\colon \mathcal{M}_{2-r}(\mathbb{R}^n) \to \mathbb{R}$, satisfying Assumption A9, such that the mapping $g\colon \mathcal{M}_{2-r}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}$ satisfies $g(\mu, x) = \delta G/\delta\mu(\mu, x)$ and, thus, $\nabla_x g(\mu, x) = \partial_\mu G(\mu, x)$.

Generally speaking, a mean field game problem is said to be variational if the associated mean field game system can be interpreted as the first-order optimality condition of a variational problem. Usually (i.e., in standard mean field games), the criterion of the variational problem involves a potential functional whose derivative –understood in a suitable sense– coincides with the interaction cost of the game (see, for instance, [26] when the mean field game is formulated as a system of PDEs, and [35, Chapter 6] for the probabilistic counterpart). Here, the mean field mapping $G\colon \mathcal{M}_{2-r}(\mathbb{R}^n) \to \mathbb{R}$ introduced in the above assumption plays the role of the potential, with the derivative understood in the flat sense for Nature and in the Lions sense for the representative player.

**Corollary 16.** *Let Assumptions A1–A5 and A10 be satisfied. Then, there exists a unique mean field game equilibrium $(\psi, q, \mu) \in \mathcal{A} \times \mathcal{Q} \times \mathcal{M}_{2-r}(\mathbb{R}^n)$, to the problem (MFG), where we recall that $r$ is defined in Assumption A2. The equilibrium is fully characterized as the solution to the system formed by $(\mathrm{Opt_N})$–$(\mathrm{Opt_C})$ with the terminal conditions in the first two systems being replaced by*

$$p_T = q_T \nabla_x g(X_T^\psi, \mu), \quad Y_T = g(X_T^\psi, \mu), \tag{82}$$

*complemented by the equilibrium condition (MFG-eq), namely $\mu = (q_T \mathbb{P})_{X_T^\psi}$.*

*Proof. Step 1: Necessary and sufficient condition for equilibrium.* Let $\mu \in \mathcal{M}_{2-r}(\mathbb{R}^n)$. Applying Corollary 15 to the parametrized mean field mapping $G[\mu]\colon \mathcal{M}_{2-r}(\mathbb{R}^n) \to \mathbb{R}$ defined as follows

$$G[\mu](\nu) = \int_{\mathbb{R}^n} g(\mu, x) \mathrm{d}\nu(x), \quad \nu \in \mathcal{M}_{2-r}(\mathbb{R}^n),$$

we deduce that the the system $(\mathrm{Opt_N})$–$(\mathrm{Opt_C})$, with the terminal conditions (82), is a necessary and sufficient condition of for equilibrium when the interaction term $\mu$ is frozen.

When complemented by the equilibrium condition $\mu = (q_T \mathbb{P})_{X_T^\psi}$, they provide a characterization of the solutions to the mean field game (MFG).

*Step 2: Uniqueness.* By Corollary 15, there exists a unique solution $(\psi, q) \in \mathcal{A} \times \mathcal{Q}$ to the system ($\mathrm{Opt_N}$)-($\mathrm{Opt_C}$) with the terminal conditions

$$p_T = q_T \nabla_x g\left((q_T\mathbb{P})_{X_T^\psi}, X_T^\psi\right), \quad Y_T = g\left((q_T\mathbb{P})_{X_T^\psi}, X_T^\psi\right).$$

This system coincides with the necessary and sufficient condition identified in the first step, which proves that there exists a unique solution to (MFG) in the space mentioned in the statement. □

**Perspectives.** We conclude this section with a brief discussion about mean field game model beyond the variational case. A natural question arises as to how one might treat mean field games that lack an underlying variational structure. The monotonicity assumptions imposed on the flat and Lions derivatives of $G$ in A9 (and thus on $g$ in A10) in the MFC problem already suggest the type of conditions that can be imposed on the interaction terms to ensure uniqueness of solutions, in the spirit of the classical Lasry–Lions monotonicity condition for standard mean field games. We refer to our companion work [49] for complete results in this direction.

# 5 Proof of Theorem 10

In this section, we establish all the intermediate results used in the proof of Theorem 10. The presentation is organized into three subsections. In Subsection 5.1, we establish the existence of a min–max solution to the problem (P'), corresponding to Step 1 and Step 2 in the proof of Theorem 10. Subsection 5.2 provides the necessary and sufficient conditions for the control problem solved by Nature, thus covering the arguments developed in Step 3 of the proof. Finally, Subsection 5.3 focuses on the central planner and forms the basis of Step 4 in the proof of Theorem 10.

## 5.1 Existence of a saddle point to (P')

This subsection is dedicated to the proof of the existence of a saddle point to the problem (P'), for given values of $c_1, c_2 > 0$. This corresponds to the first step in the proof of Theorem 10. Without any loss of generality, we can assume that

$$c_1 > \mathbb{E}\int_0^T q_t^0 f(t,0,0)\mathrm{d}t, \tag{83}$$

where $\bar{q}^0$ denotes the solution of

$$\mathrm{d}q_t^0 = q_t^0 \partial_y f(t,0,0)\mathrm{d}t + q_t^0 \partial_z f(t,0,0)\cdot \mathrm{d}W_t, \quad t \in [0,T]. \tag{84}$$

We notice that the right-hand side on (83) is equal to $\mathcal{S}(q^0)$. Indeed

$$\mathcal{S}(q^0) = \mathbb{E}\int_0^T q_t^0 f^\star\left(t, \partial_y f(t,0,0), \partial_z f(t,0,0)\right)\mathrm{d}t = -\mathbb{E}\int_0^T q_t^0 f(t,0,0)\mathrm{d}t.$$

The purpose is thus to establish the following statement:

**Lemma 17.** *There exists a solution* $(\psi, q) \in \mathcal{A}_{c_2} \times \mathcal{Q}_{c_1}$ *to* (P').

Before we provide a sketch of the proof of this result, we introduce a variant of the Nature optimization problem. Existence of a saddle point is proven by means of weak compactness arguments (in $L^p$ spaces), which are developed in this subsection. In this regard, the nonlinear form of the state equation (4) causes additional difficulties, as the product form of the coefficients is not appropriate for weak convergence arguments. For this reason, it is easier to consider weak limits of the two products $(q_t Y_t^\star)_{t\in[0,T]}$ and $(q_t Z_t^\star)_{t\in[0,T]}$, each being viewed as a single process. However, this makes more difficult the identification of the limit points as solutions of an equation of the form (4) (because weak limits of the products must be shown to have a product form, which writing may be difficult to establish if the weak limit of $(q_t)_{t\in[0,T]}$ vanishes). This prompts us to introduce a variant of the problem (P'), and in particular to define the perspective function $qf^\star(\omega, t, y^\star/q, z^\star/q)$ for any $(\omega, t, q, y^\star, z^\star) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ of $f^\star$ (see [15, 44] for a presentation) with respect to its last two variables. We further introduce its lower semi-continuous envelope (or its bidual) $\tilde{f}^\star \colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$,

$$\tilde{f}^\star(\omega, t, q, y^\star, z^\star) = \begin{cases} qf^\star\left(\omega, t, \frac{y^\star}{q}, \frac{z^\star}{q}\right), & q > 0, \\ \text{rec}f^\star(\omega, t, q, y^\star, z^\star), & q = 0, \\ +\infty, & \text{q} < 0. \end{cases} \tag{85}$$

Here $\text{rec}f^\star(\omega, t, q, \cdot, \cdot)$ denotes the recession function of $f^\star(\omega, t, q, \cdot, \cdot)$ (with respect to the last two variables of $f^\star$). By [15, Lemma 1.156], it coincides with the support function of $f^\star(\omega, t, 0, \cdot, \cdot)$, i.e.,

$$\text{rec}f^\star(\omega, t, 0, y^\star, z^\star) = \sup_{(y,z)\in\mathbb{R}\times\mathbb{R}^d} \left\{ \langle y, y^\star \rangle + \langle z, z^\star \rangle, \ f(\omega, t, 0, y, z) < +\infty \right\}.$$

Because $f(\omega, t, q, \cdot, \cdot)$ has full support, the recession function at $q = 0$ is given by

$$\text{rec}f^\star(\omega, t, q, y^\star, z^\star) = \begin{cases} 0, & (y^\star, z^\star) = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Finally, the function $\tilde{f}^\star$ is equal to

$$\tilde{f}^\star(\omega, t, q, y^\star, z^\star) = \begin{cases} qf^\star\left(\omega, t, \frac{y^\star}{q}, \frac{z^\star}{q}\right), & q > 0, \\ 0, & (q, y^\star, z^\star) = 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{86}$$

For simplicity, we call $\tilde{f}^\star$ the perspective function of $f^\star$ when there is no ambiguity. Because $f^\star(\omega, t, \cdot, \cdot)$ is convex and lower semi-continuous, its perspective function $\tilde{f}^\star$ is convex with respect to its three last variables and lower semi-continuous. For any non-negative valued Itô process $q = (q_t)_{t\in[0,T]}$, satisfying $\mathbb{E}[q_T^*] < +\infty$, and admitting the expansion

$$\mathrm{d}q_t = \tilde{Y}_t^\star \mathrm{d}t + \tilde{Z}_t^\star \cdot \mathrm{d}W_t, \quad t \in [0,T], \tag{87}$$

for some (uniquely defined) $\mathbb{F}$-progressively measurable process $\tilde{Y}^\star = (\tilde{Y}_t^\star)_{t\in[0,T]}$ and $\tilde{Z}^\star = (\tilde{Z}_t^\star)_{t\in[0,T]}$, with values in $\mathbb{R}$ and $\mathbb{R}^d$ respectively and satisfying

$$\mathbb{P}\left(\left\{\int_0^T (|\tilde{Y}_t^\star| + |\tilde{Z}_t^\star|^2)\mathrm{d}t < +\infty\right\}\right) = 1, \tag{88}$$

we define the perspective generalized entropy of $q$ by letting

$$\tilde{\mathcal{S}}(q) := \mathbb{E}\left[\int_0^T \tilde{f}^\star(t, q_t, \tilde{Y}_t^\star, \tilde{Z}_t^\star)\mathrm{d}t\right]. \tag{89}$$

Recalling the lower bound (16) and using the fact that $\mathbb{E}[q_T^*] < +\infty$, we notice that the expectation right above is well-defined; it belongs to $(-\infty, +\infty]$. And then, we introduce the perspective min-max problem

$$\sup_{q\in\tilde{\mathcal{Q}}_{c_1}} \inf_{\psi\in\mathcal{A}_{c_2}} \tilde{\mathcal{J}}(q,\psi), \tag{$\tilde{\text{P}}$'}$$

where the mapping $\tilde{\mathcal{J}}$ is given by

$$\tilde{\mathcal{J}}(q,\psi) := \mathcal{R}(q,\psi) - \tilde{\mathcal{S}}(q); \tag{90}$$

recall (1) for the definition of $\mathcal{R}$. Above, the set $\tilde{\mathcal{Q}}_{c_1}$ is defined as the collection of $q \in \tilde{\mathcal{Q}}$ such that $\tilde{\mathcal{S}}(q) \leq c_1$, where $\tilde{\mathcal{Q}}$ is the set of non-negative valued measurable Itô processes $q = (q_t)_{t\in[0,T]}$ satisfying (87), such that $q_0 = 1$, $\mathbb{E}[q_T^*] \leq \exp(\alpha T)$, and $\tilde{\mathcal{S}}(q) < +\infty$.

Since we restricted the controlled dynamics (4) to processes $(q_t)_{t\in[0,T]}$ that do not vanish, it is easy to see that any $q \in \mathcal{Q}_{c_1}$ belongs to $\tilde{\mathcal{Q}}_{c_1}$. Indeed, $\mathcal{S}(q)$ and $\tilde{\mathcal{S}}(q)$ coincide in this setting. Moreover, the bound $\mathbb{E}[q_T^*] \leq \exp(\alpha T)$ follows from the facts that $Y^\star$ is bounded by $\alpha$ and $(\mathcal{E}_t(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s))_{t\in[0,T]}$ is a martingale, see Lemma 9.

Existence of a saddle point to ($\tilde{\text{P}}$') is established in the next subsection; see Lemma 23. Taking the latter for granted, Lemma 17 can be derived as follows:

*Proof of Lemma 17.* Since existence of a saddle point to ($\tilde{\text{P}}$') is provided by Lemma 23, it suffices to show that any solution to ($\tilde{\text{P}}$') is a solution to (P'). Let $(\bar{q}, \bar{\psi})$ be a solution to ($\tilde{\text{P}}$'), that is to say

$$\tilde{\mathcal{J}}(q,\bar{\psi}) \leq \tilde{\mathcal{J}}(\bar{q},\bar{\psi}) \leq \tilde{\mathcal{J}}(\bar{q},\psi), \quad \forall (q,\psi) \in \tilde{\mathcal{Q}}_{c_1} \times \mathcal{A}_{c_2}. \tag{91}$$

By Lemma 18 (which is stated and proven in Subsection 5.1.1 below), the process $\bar{q}$ is positive in the sense that $\mathbb{P}(\{\inf_{t\in[0,T]} \bar{q}_t > 0\}) = 1$. This makes it possible to let $(Y_t^\star := \tilde{Y}_t^\star/\bar{q}_t)_{t\in[0,T]}$ and $(Z_t^\star = \tilde{Z}_t^\star/\bar{q}_t)_{t\in[0,T]}$, from which we deduce

$$\mathrm{d}\bar{q}_t = \bar{q}_t Y_t^\star \mathrm{d}t + \bar{q}_t Z_t^\star \cdot \mathrm{d}W_t, \quad t \in [0,T]; \quad \bar{q}_0 = 1.$$

By definition of the perspective generalized entropy

$$\mathcal{S}(\bar{q}) = \tilde{\mathcal{S}}(\bar{q}) \leq c_1,$$

which proves that $\bar{q}$ belongs to $\mathcal{Q}_{c_1}$. Then, by the definition (90) of $\tilde{\mathcal{J}}$, by the optimality condition (91) and since $\tilde{\mathcal{Q}}_{c_1}$ contains $\mathcal{Q}_{c_1}$, we have

$$\mathcal{J}(q,\bar{\psi}) \leq \mathcal{J}(\bar{q},\bar{\psi}) \leq \mathcal{J}(\bar{q},\psi), \quad \forall (q,\psi) \in \mathcal{Q}_{c_1} \times \mathcal{A}_{c_2},$$

concluding the proof. □

#### 5.1.1 Trajectories of the perspective problem and positivity of the optimal ones

The third item in the following lemma was used in the proof of Lemma 17. The first two items are also used in the proof of Lemma 23.

**Lemma 18.** *Let* $q \in \tilde{\mathcal{Q}}_{c_1}$ *and* $(\tilde{Y}^\star, \tilde{Z}^\star)$ *be as in the representation* (87).

*i. Letting*

$$Y_t^\star = \mathbb{1}_{\{q_t>0\}} \frac{\tilde{Y}_t^\star}{q_t}, \quad Z_t^\star = \mathbb{1}_{\{q_t>0\}} \frac{\tilde{Z}_t^\star}{q_t}, \quad t \in [0,T], \tag{92}$$

*it holds*

$$\mathbb{P} \otimes \mathrm{Leb}_{[0,T]} \left( \{(\omega,t) \in \Omega \times [0,T],\ |Y_t^\star| > \alpha\} \right) = 0, \tag{93}$$

*and* $q$ *can be expanded as*

$$\mathrm{d}q_t = q_t Y_t^\star \mathrm{d}t + q_t Z_t^\star \cdot \mathrm{d}W_t, \quad t \in [0,T]. \tag{94}$$

*ii. Moreover, letting* $\tau := \inf\{t \in [0,T],\ q_t = 0\}$ *(with* $\inf \emptyset = +\infty$*), it also holds* $\mathbb{P}(\{\sup_{t\in[\tau,T]} q_t > 0\} \cap \{\tau < T\}) = 0$ *(i.e., 0 is an absorbing state). And then,*

$$\tilde{\mathcal{S}}(q) = \mathbb{E}\left[\int_0^T \tilde{f}^\star(t, q_t, \tilde{Y}_t^\star, \tilde{Z}_t^\star)\mathrm{d}t\right] = \mathbb{E}\left[\int_0^\tau q_t f^\star(t, Y_t^\star, Z_t^\star)\mathrm{d}t\right]. \tag{95}$$

*We also have* $\mathbb{E}[q_T^*] < +\infty$ *and there exists a constant* $C$*, which depends on* $q$ *only via* $c_1$*, such that*

$$\mathbb{E}\left[\left(\int_0^T |\tilde{Z}_s^\star|^2 \mathrm{d}s\right)^{1/2}\right] \le C. \tag{96}$$

*iii. Lastly, if for a certain* $\psi \in \mathcal{A}_{c_2}$*, the pair* $(\psi, q) \in \mathcal{A}_{c_2} \times \tilde{\mathcal{Q}}_{c_1}$ *is a solution to* ($\tilde{\mathrm{P}}$'). *Then* $\mathbb{P}(\{\inf_{t\in[0,T]} q_t > 0\}) = 1$*, and (in particular)* $q \in \mathcal{Q}_{c_1}$.

*Remark* 19. The following two comments are in order:

1. In dimension $d = 1$, the CIR model, i.e.,

$$\mathrm{d}q_t = \sqrt{q_t}\mathrm{d}W_t, \quad t \in [0,\tau),$$

   provides an interesting example in which $q$ may vanish even if the entropy, which is here equal to $\mathbb{E}[\int_0^{\tau\wedge T} q_t^{-1} q_t \mathrm{d}t] = \mathbb{E}[\tau \wedge T]$, is finite.

2. When $q$ vanishes, it does not make sense to represent it in the form of a (weighted) Doléans-Dade exponential martingale. This observation causes additional difficulites in the analysis.

*Proof. Step 1: Representation of* $q \in \tilde{\mathcal{Q}}_{c_1}$. We recall that $q$ can be represented as

$$q_t = 1 + \int_0^t \tilde{Y}_s^\star \mathrm{d}s + \int_0^t \tilde{Z}_s^\star \cdot \mathrm{d}W_s, \quad t \in [0,T],$$

with

$$\tilde{\mathcal{S}}(q) = \mathbb{E}\left[\int_0^T \tilde{f}^\star(t, q_t, \tilde{Y}_t^\star, \tilde{Z}_t^\star)\mathrm{d}t\right] \in (-\infty, c_1].$$

Using (16) together with the bound $\mathbb{E}[q_T^*] \le \exp(\alpha T)$, we deduce that

$$\mathbb{E}\left[\int_0^T \mathbb{1}_{\{q_t>0\}} q_t \left(\chi_{\mathcal{B}}\left(\frac{\tilde{Y}_t^\star}{\alpha q_t}\right) + \frac{1}{2\beta}\left|\frac{\tilde{Z}_t^\star}{q_t}\right|^2\right) \mathrm{d}t\right] < +\infty. \tag{97}$$

This proves in particular that

$$\mathbb{P} \otimes \mathrm{Leb}_{[0,T]}\left(\left\{(\omega, t) \in \Omega \times [0,T],\ q_t > 0, \quad |\tilde{Y}_t^\star| > \alpha q_t\right\}\right) = 0. \tag{98}$$

Moreover, recalling the definition (86) of $\tilde{f}^\star$, we also have

$$\mathbb{P} \otimes \mathrm{Leb}_{[0,T]}\left(\left\{(\omega, t) \in \Omega \times [0,T],\ q_t = 0,\ |\tilde{Y}_t^\star| + |\tilde{Z}_t^\star| > 0\right\}\right) = 0. \tag{99}$$

With the notation (92), (93) and (94) easily follow.

*Step 2: Proving that $q$ stays in 0 once it has touched it.* Recall that $\tau := \inf\{t \in [0,T],\ q_t = 0\}$ (with $\inf \emptyset = +\infty$). We want to prove that $\mathbb{P}(\{\sup_{t\in[\tau,T]} q_t > 0\} \cap \{\tau < T\}) = 0$. The proof is as follows. For any $\epsilon > 0$, let $\varrho^\epsilon := \inf\{t \in [\tau, T],\ q_t = \epsilon\}$, with the convention that $\varrho^\epsilon = +\infty$ if $\tau = +\infty$ or if $\tau \le T$ and $q$ does not touch $\epsilon$ between $\tau$ and $T$. Using (93) and (94), we then notice that

$$\mathrm{d}\left(\exp(\alpha t) q_t\right) \ge \exp(\alpha t) q_t Z_t^\star \cdot \mathrm{d}W_t, \quad t \in [0,T].$$

By localization (use (88) together with the fact that $\tilde{Z}_t^\star = q_t Z_t^\star$), we can find a non-decreasing sequence of stopping times $(\sigma_k)_{k\ge1}$, converging to $T$ (almost surely), such that

$$\forall k \ge 1, \quad \mathbb{E}\left[\int_0^{\sigma_k} q_t^2 |Z_t^\star|^2 \mathrm{d}t\right] < +\infty.$$

And then,

$$\exp\left(\alpha\tau \wedge \sigma_k\right) q_{\tau\wedge\sigma_k} \ge \mathbb{E}\left[\exp\left(\alpha\varrho^\epsilon \wedge \sigma_k\right) q_{\varrho^\epsilon\wedge\sigma_k} | \mathcal{F}_\tau\right].$$

Letting $k$ tend to $+\infty$ and using a conditional version of Fatou's lemma, we obtain, $\mathbb{P}$-almost surely,

$$\exp(\alpha\tau \wedge T) q_{\tau\wedge T} \ge \mathbb{E}\left[\exp(\alpha\varrho^\epsilon \wedge T) q_{\varrho^\epsilon\wedge T} | \mathcal{F}_\tau\right].$$

We deduce that there exists a constant $c > 0$, only depending on $T$ and $\alpha$, such that

$$q_{\tau\wedge T} \ge c\mathbb{E}\left[q_{\varrho^\epsilon\wedge T} | \mathcal{F}_\tau\right].$$

Multiply both sides by $\mathbb{1}_{\{\tau<T\}}$ and take expectation under $\mathbb{P}$. Since $q_\tau = 0$ when $\tau < T$, we get

$$\mathbb{E}\left[\mathbb{1}_{\{\tau<T\}} q_{\varrho^\varepsilon\wedge T}\right] = 0.$$

This shows $\mathbb{P}(\{\tau < T\} \cap \{\varrho^\epsilon \le T\}) = 0$. Letting $\epsilon$ tend to 0, we derive the expected claim, that is $\mathbb{P}(\{\tau < T\} \cap \{\sup_{t\in[\tau,T]} q_t > 0\}) = 0$. This makes it possible to prove (95). Indeed, together with (99), we obtain

$$\mathbb{P} \otimes \mathrm{Leb}_{[0,T]}\left(\left\{(\omega, t) \in \Omega \times [0,T],\ t > \tau,\ |\tilde{Y}_t^\star| + |\tilde{Z}_t^\star| > 0\right\}\right) = 0,$$

from which we deduce that (recalling (85))

$$\mathbb{E}\left[\int_{\tau}^{T} \tilde{f}^{\star}(t, q_t, \tilde{Y}_t^{\star}, \tilde{Z}_t^{\star})\mathrm{d}t\right] = 0.$$

Identity (95) easily follows.

We now prove that $\mathbb{E}[q_T^*] < +\infty$. Letting $(\tilde{q}_t := q_t \exp(-\int_0^t Y_s^{\star}\mathrm{d}s))_{t\in[0,T]}$, we deduce from (94) that

$$\mathrm{d}\tilde{q}_t = \tilde{q}_t Z_t^{\star} \cdot \mathrm{d}W_t, \quad t \in [0,T].$$

By Itô's formula,

$$\mathrm{d}\left[\tilde{q}_t \ln(\tilde{q}_t)\right] = \frac{1}{2}\tilde{q}_t |Z_t^{\star}|^2 \mathrm{d}t + \left[\ln(\tilde{q}_t) + 1\right]\tilde{q}_t Z_t^{\star} \cdot \mathrm{d}W_t, \quad t \in [0,\tau).$$

By a localization argument (together with (97)), we deduce that $\mathbb{E}[\tilde{q}_\tau \ln(\tilde{q}_\tau)] < +\infty$. And then, by $L\log(L)$-Doob's maximal inequality, we obtain $\mathbb{E}[\tilde{q}_T^*] = \mathbb{E}[\sup_{t\in[0,\tau]} \tilde{q}_t] \leq C$, for a constant $C$ that depends on $q$ only via $c_1$. We deduce that $\mathbb{E}[q_T^*] < C\exp(\alpha T)$. By Burkholder-Davis-Gundy inequalities, (96) easily follows.

*Step 3: Contradicting the fact that $\tau \leq T$, when $(\psi, q)$ is a saddle-point.* We now prove the final result, that is $\mathbb{P}(\{\tau \leq T\}) = 0$ when $q$ satisfies the optimality property of a saddle-point. For $\theta \in (0,1)$, we let $q^\theta := \theta q + (1-\theta)q^0$, where we recall (84) for the definition of $q^0$. We have

$$\begin{aligned}
\mathrm{d}q_t^\theta &= \left[\theta q_t Y_t^{\star} + (1-\theta)q_t^0 \partial_y f(t,0,0)\right]\mathrm{d}t + \left[\theta q_t Z_t^{\star} + (1-\theta)q_t^0 \partial_z f(t,0,0)\right] \cdot \mathrm{d}W_t \\
&=: \tilde{Y}_t^{\star,\theta}\mathrm{d}t + \tilde{Z}_t^{\star,\theta} \cdot \mathrm{d}W_t,
\end{aligned}$$

for any $t \in [0,T]$. Since $q^\theta$ is positive valued, for $\theta \in [0,1)$, we can let $Y_t^{\star,\theta} = \tilde{Y}_t^{\star,\theta}/q_t^\theta$ and $Z_t^{\star,\theta} = \tilde{Z}_t^{\star,\theta}/q_t^\theta$, for $t \in [0,T]$. By (83), we know that $\tilde{\mathcal{S}}(q^0) = \mathcal{S}(q^0) < c_1$. By convexity of $\tilde{\mathcal{S}}$ (see Step 2 in the proof of Proposition 20), we deduce that $\tilde{\mathcal{S}}(q^\theta) \leq c_1$, and then $q^\theta \in \tilde{\mathcal{Q}}_{c_1}$ for all $\theta \in [0,1]$.

By definition of $\tilde{\mathcal{J}}$ (see (89) and (90)),

$$\tilde{\mathcal{J}}(q^\theta, \psi) = \mathcal{G}(q_T^\theta, X_T^\psi) + \mathbb{E}\left[\int_0^T q_s^\theta \ell(s,\psi_s)\mathrm{d}s\right] - \mathbb{E}\left[\int_0^T \tilde{f}^{\star}\left(s, q_s^\theta, \tilde{Y}_s^{\star,\theta}, \tilde{Z}_s^{\star,\theta}\right)\mathrm{d}s\right].$$

We then subtract $\tilde{\mathcal{J}}(q,\psi)$ on both sides. Recalling the definition (86) of $\tilde{f}^{\star}$ we obtain

$$\begin{aligned}
\tilde{\mathcal{J}}(q^\theta, \psi) - \tilde{\mathcal{J}}(q,\psi) = &\left(\mathcal{G}(q_T^\theta, X_T^\psi) - \mathcal{G}(q_T, X_T^\psi)\right) + \mathbb{E}\left[\int_0^T \left(q_s^\theta - q_s\right)\ell(s,\psi_s)\mathrm{d}s\right] \\
&- \mathbb{E}\left[\int_0^T \left(\tilde{f}^{\star}\left(s, q_s^\theta, \tilde{Y}_s^{\star,\theta}, \tilde{Z}_s^{\star,\theta}\right) - \tilde{f}^{\star}\left(s, q_s, \tilde{Y}_s^{\star}, \tilde{Z}_s^{\star}\right)\right)\mathrm{d}s\right] \\
=: &S_1^\theta + S_2^\theta - S_3^\theta.
\end{aligned}$$

Using the fact that $q_t^\theta - q_t = (1-\theta)(q_t^0 - q_t)$ together with the regularity of $\mathcal{G}$ in the variable $q$ and the integrability properties of $\psi$, and then applying Lemma 40, we deduce that there exists a positive constant $C$ such that, for any $\theta \in (0,1)$,

$$|S_1^\theta| + |S_2^\theta| \leq C(1-\theta).$$

Similarly, by strong convexity of $f^\star$ (in the last variable), see Remark 2, there exists $c > 0$ (independent of $\theta$) such that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \tilde{f}^\star\left(s, q_s^\theta, \tilde{Y}_s^{\star,\theta}, \tilde{Z}_s^{\star,\theta}\right) \mathrm{d}s\right] \\
&= \mathbb{E}\left[\int_0^T q_s^\theta f^\star\left(s, \frac{\theta q_s}{q_s^\theta} Y_s^{\star,\theta} + \frac{(1-\theta)q_s^0}{q_s^\theta}\partial_y f_s^0, \frac{\theta q_s}{q_s^\theta} Z_s^{\star,\theta} + \frac{(1-\theta)q_s^0}{q_s^\theta}\partial_z f_s^0\right) \mathrm{d}s\right] \\
&\leq \mathbb{E}\left[\int_0^T \left(\theta \tilde{f}^\star\left(s, q_s, \tilde{Y}_s^\star, \tilde{Z}_s^\star\right) + (1-\theta) q_s^0 f_s^0 - c(1-\theta)\frac{\theta q_s q_s^0}{q_s^\theta}|Z_s^\star - \partial_z f_s^0|^2\right) \mathrm{d}s\right],
\end{aligned}$$

where we have used the shorthand notations $\partial_y f_s^0 := \partial_y f(s,0,0)$ and $\partial_z f_s^0 := \partial_z f(s,0,0)$, and the duality identity $f^\star(s, \partial_y f_s^0, \partial_z f_s^0) = f(s,0,0) = f_s^0$. The last line, together with the fact that $\|f^0\|_{L^\infty(\mathbb{F})} < +\infty$ and $\|\partial_z f^0\|_{L^\infty(\mathbb{F})} < +\infty$ and the standard inequality $|Z_s^\star - \partial_z f_s^0|^2 \geq \frac{1}{2}|Z_s^\star|^2 - |\partial_z f_s^0|^2$, yields the following lower bound

$$-S_3^\theta \geq (1-\theta)\left(-C - \mathbb{E}\left[\int_0^T \tilde{f}^\star\left(s, q_s, \tilde{Y}_s^\star, \tilde{Z}_s^\star\right) \mathrm{d}s\right] + \frac{c}{2}\mathbb{E}\left[\int_0^T \frac{\theta q_s q_s^0}{q_s^\theta}|Z_s^\star|^2 \mathrm{d}s\right]\right).$$

Using the fact that $\tilde{\mathcal{S}}(q) \leq c_1$, we deduce in the end that (for a possibly new value of $C$)

$$\tilde{\mathcal{J}}(q^\theta, \psi) - \tilde{\mathcal{J}}(q, \psi) \geq (1-\theta)\left(-C + \frac{c}{2}\mathbb{E}\left[\int_0^T \frac{\theta q_s q_s^0}{q_s^\theta}|Z_s^\star|^2 \mathrm{d}s\right]\right). \tag{100}$$

It then remains to observe that (whether the right-hand side is finite or not)

$$\lim_{\theta \to 1} \mathbb{E}\left[\int_0^T \frac{\theta q_s q_s^0}{q_s^\theta}|Z_s^\star|^2 \mathrm{d}s\right] = \mathbb{E}\left[\int_0^\tau q_s^0 |Z_s^\star|^2 \mathrm{d}s\right]. \tag{101}$$

Since $q$ is an optimizer of $\tilde{\mathcal{J}}(\cdot, \psi)$ over $\tilde{\mathcal{Q}}_{c_1}$, it holds $\tilde{\mathcal{J}}(q^\theta, \psi) \leq \tilde{\mathcal{J}}(q, \psi)$ implying that the left-hand side in (100) is non-positive. Combining the last two lines, this shows that the right-hand side on the above identity is necessarily finite. We claim that this implies that $\mathbb{P}(\{\tau \leq T\}) = 0$.

Assume by a way of contradiction that $\mathbb{P}(\{\tau \leq T\}) > 0$. For $t \in [0, \tau)$, we can expand $\ln(q_t)$ by means of Itô's formula. We get

$$\begin{aligned}
\mathrm{d}\ln(q_t) &= \left(Y_t^\star - \frac{1}{2}|Z_t^\star|^2\right) \mathrm{d}t + Z_t^\star \cdot \mathrm{d}W_t \\
&= \left(Y_t^\star - \frac{1}{2}|Z_t^\star|^2 + Z_t^\star \cdot \partial_z f_t^0\right) \mathrm{d}t + Z_t^\star \cdot \mathrm{d}\left(W_t - \int_0^t \partial_z f_s^0 \mathrm{d}s\right), \quad t \in [0, \tau).
\end{aligned}$$

Let $\mathbb{Q}^0 := \mathcal{E}_T(\int_0^\cdot \partial_z f_s^0 \cdot \mathrm{d}W_s)$. Setting $\sigma^\epsilon := \inf\{t \geq 0,\ q_t \leq \epsilon\}$ for any $\epsilon \in (0,1)$ (with the convention that $\inf \emptyset = +\infty$), we have

$$\begin{aligned}
-\mathbb{E}^{\mathbb{Q}^0}\left[\ln(q_{\sigma^\epsilon \wedge T})\right] &\leq \frac{1}{2}\mathbb{E}^{\mathbb{Q}^0}\left[\int_0^T |Z_t^\star|^2 \mathrm{d}t\right] + \mathbb{E}^{\mathbb{Q}^0}\left[\int_0^T |Z_t^\star||\partial_z f_t^0| \mathrm{d}t\right] + \alpha T, \\
&\leq \mathbb{E}^{\mathbb{Q}^0}\left[\int_0^T |Z_t^\star|^2 \mathrm{d}t\right] + C,
\end{aligned}$$

for a constant $C$ depending on $\alpha$, $T$ and $\|\partial_z f^0\|_{L^\infty(\mathbb{F})}$. Now, if $\mathbb{P}(\{\tau \leq T\}) > 0$, then $\mathbb{Q}^0(\{\tau \leq T\}) > 0$, and $\sup_{\epsilon > 0}[-\ln(\epsilon)\mathbb{Q}^0(\{\sigma^\epsilon \leq T\})] = +\infty$, and thus the right-hand side is also infinite, which contradicts (100) and (101). □

#### 5.1.2 Solvability of the perspective min-max problem

**Proposition 20.** *Let $\psi \in L^\infty(\mathbb{F}, \mathbb{R}^n)$ and $c_1 > 0$. Viewing $\tilde{\mathcal{Q}}_{c_1}$ as a subset of $L^1(\Omega \times [0,T], \mathbb{P} \otimes \mathrm{Leb}_{[0,T]})$ equipped with the weak topology $\sigma(L^1, L^\infty)$, $\tilde{\mathcal{Q}}_{c_1}$ is (weakly) compact and convex, and satisfies*

$$\sup_{q \in \tilde{\mathcal{Q}}_{c_1}} \sup_{t \in [0,T]} \mathbb{E}[h(q_t)] < +\infty. \tag{102}$$

*In addition, the mapping $\tilde{\mathcal{Q}}_{c_1} \ni q \mapsto \tilde{\mathcal{J}}(\psi, q)$ is strictly concave and upper semi-continuous (w.r.t. the weak topology).*

*Remark* 21. As a corollary of the proof, we obtain that the functional $\tilde{\mathcal{S}}$ is convex, which has further applications. Indeed, for any $\theta \in [0,1]$, let (as in the proof of Lemma 18) $q^\theta := (1-\theta)q^0 + \theta q$, where $q^0$ is defined as in (84). We observe that $q^\theta$ is positive valued for each $\theta \in [0,1)$. By Lemma 18, it is easy to see that, for every $\theta \in [0,1)$, $q^\theta \in \mathcal{Q}$. Moreover, by convexity of $\tilde{\mathcal{Q}}_{c_1}$, we have $q^\theta \in \mathcal{Q}_{c_1}$, provided that $c_1$ is large enough, which is not a restriction here. This shows that $q^\theta \in \mathcal{Q}_{c_1}$ and, more generally, that $\mathcal{S}(q^\theta) \le (1-\theta)\mathcal{S}(q^0) + \theta \tilde{\mathcal{S}}(q)$. Since $\mathcal{S}(q^0) \in \mathbb{R}$, we deduce that

$$\limsup_{\theta \to 1} \mathcal{S}(q^\theta) \le \tilde{\mathcal{S}}(q).$$

This observation allows us to extend results holding on $\mathcal{Q}_{c_1}$ to $\tilde{\mathcal{Q}}_{c_1}$, such as the duality inequality (14), and Lemmas 40 and 41. Similarly, Lemmas 43 and 44, which are invoked in Step 5 of the proof below, can also be extended to sequences with values in $\tilde{\mathcal{Q}}$, using the additional fact that the function $h$ is convex (recall (10) for the definition of $h$).

*Proof.* The proof is divided into six steps. In Step 1, we show the relative weak compactness of $\tilde{\mathcal{Q}}_{c_1}$. In Step 2, we establish the (strict) concavity of the mapping $\tilde{\mathcal{J}}$ and we prove the convexity of $\tilde{\mathcal{Q}}_{c_1}$. In Steps 3 and 4, we show the weak lower semi-continuity of $\tilde{\mathcal{S}}$ and the weak compactness of $\tilde{\mathcal{Q}}_{c_1}$. In Step 5, we establish the weak continuity of $\mathcal{R}$. In Step 6, we conclude the proof. Throughout, the value of $\psi$ is fixed. For this reason, we omit it in many notations. For instance, we just write $\tilde{\mathcal{J}}(q)$ for $\tilde{\mathcal{J}}(q, \psi)$.

*Step 1: relative weak compactness of $\tilde{\mathcal{Q}}_{c_1}$.* We first establish the relative weak compactness, in $L^1(\Omega \times [0,T], \mathbb{P} \otimes \mathrm{Leb}_{[0,T]})$ equipped with the weak topology $\sigma(L^1, L^\infty)$, of any subset $\mathcal{D}$ of $L \log L(\mathbb{F})$ that is bounded in the sense that

$$\sup_{q \in \mathcal{D}} \sup_{t \in [0,T]} \mathbb{E}\big[h(q_t)\big] < +\infty.$$

The argument is classic and goes as follows. By de la Vallée Poussin Theorem [47, Theorem VI] (see also [51, Theorem 22, page 24-II]), the set $\mathcal{D}$ is uniformly integrable on $\Omega \times [0,T]$ equipped with $\mathbb{P} \otimes \mathrm{Leb}_{[0,T]}$. Then, by the Dunford-Pettis Theorem [21, Theorem 4.30], the set is weakly relatively compact. The set is weakly sequentially relatively compact by the Eberlein-Šmulian Theorem [54, Section V.6.1, p.430]. The last two statements can also be found combined into a single statement, see [51, Theorem 25, page 27-II].

We now prove that $\tilde{Q}_{c_1}$ is a bounded subset of $L \log L(\mathbb{F})$ (which corresponds to (102) in the statement). Indeed, for any $q \in \tilde{\mathcal{Q}}_{c_1}$, letting $\tau := \inf\{t \in [0,T],\ q_t = 0\}$, we know from Lemma 18 that $q$ stays equal to 0 after $\tau$. Moreover, it satisfies (95) (under the notation (93)), from which we deduce that

$$c_1 \geq \tilde{\mathcal{S}}(q) = \mathbb{E}\left[\int_0^\tau q_t f^\star(t, Y_t^\star, Z_t^\star)\mathrm{d}t\right] \geq -C + \frac{1}{2\beta}\mathbb{E}\left[\int_0^\tau q_t |Z_t^\star|^2 \mathrm{d}t\right], \tag{103}$$

where the process $(Y_t^\star, Z_t^\star)_{t\in[0,T]}$ denotes $(\mathbb{1}_{\{q_t>0\}}\tilde{Y}_t^\star/q_t, \mathbb{1}_{\{q_t>0\}}\tilde{Z}_t^\star/q_t)_{t\in[0,T]}$, and where we used the coercivity inequality (16) of $f^\star$. The constant $C$ only depends on the $L^\infty$ bound for $f^0$.

We now expand $(h(q_t))_{t\in[0,\tau)}$ by means of Itô's formula. We get

$$\mathrm{d}h(q_t) = \left((h(q_t) + q_t)Y_t^\star + \frac{1}{2}q_t|Z_t^\star|^2\right)\mathrm{d}t + (h(q_t) + q_t)Z_t^\star \cdot \mathrm{d}W_t, \quad t \in [0, \tau). \tag{104}$$

Recalling that $Y^\star$ is bounded by $\alpha$ and using a standard localization argument, we deduce that

$$\sup_{t\in[0,T]} \mathbb{E}\left[h(q_{t\wedge\tau})\right] \leq C + C\mathbb{E}\left[\int_0^\tau q_t|Z_t^\star|^2 \mathrm{d}t\right],$$

for a constant $C$ independent of $q$. Observing that $h(0) = 0$, the left-hand side is also equal to $\sup_{t\in[0,T]}\mathbb{E}[h(q_t)]$. Back to (103), we deduce that $\tilde{\mathcal{Q}}_{c_1}$ is a bounded subset of $L\log L(\mathbb{F})$.

*Step 2: (strict) concavity of $\tilde{\mathcal{Q}} \ni q \mapsto \tilde{\mathcal{J}}(q)$.* Recalling the definition of $\tilde{\mathcal{Q}}$ on the line below (90), we first notice that $\tilde{\mathcal{Q}}$ is convex. Indeed, the decomposition (87) is linear and thus stable by convex combinations. Moreover, the non-negativity constraint and the bound $\mathbb{E}[q_T^*] \leq \exp(\alpha T)$, which are both required in the definition of $\tilde{\mathcal{Q}}$, are also stable by convex combinations. It remains to see that, for any $\theta \in (0,1)$, the convex combination $q^\theta := \theta q^1 + (1-\theta)q^2$ of any two $q^1, q^2 \in \tilde{\mathcal{Q}}$ satisfies $\tilde{\mathcal{S}}(q^\theta) < +\infty$. Denoting by $(\tilde{Y}^{\star,\theta}, \tilde{Z}^{\star,\theta})$, $(\tilde{Y}^{\star,1}, \tilde{Z}^{\star,1})$ and $(\tilde{Y}^{\star,2}, \tilde{Z}^{\star,2})$ the respective representation processes of $q^\theta$, $q^1$ and $q^2$ in (87), we have

$$\tilde{\mathcal{S}}(q^\theta) = \mathbb{E}\left[\int_0^T \tilde{f}^\star(s, q_s^\theta, Y_s^{\star,\theta}, Z_s^{\star,\theta})\mathrm{d}s\right],$$

with $(\tilde{Y}^{\star,\theta}, \tilde{Z}^{\star,\theta}) = \theta(\tilde{Y}^{\star,1}, \tilde{Z}^{\star,1}) + (1-\theta)(\tilde{Y}^{\star,2}, \tilde{Z}^{\star,2})$. Since $f^\star$ is strictly convex in its last two variables, see Remark 2, $\tilde{f}^\star$ is (strictly) jointly convex in its last three arguments, see [15, Lemma 1.157]. It easily follows that $\tilde{S}(q^\theta) < +\infty$, i.e., $\tilde{\mathcal{Q}}$ is convex. Moreover, $\tilde{\mathcal{S}}$ is strictly convex on $\tilde{\mathcal{Q}}$.

It remains to see that, by the concavity Assumption A8, $\tilde{\mathcal{Q}} \ni q \mapsto \mathcal{R}(q)$ defined in (1) is concave (with the shorthand notation $\mathcal{R}(q)$ for $\mathcal{R}(q,\psi)$). We deduce that $\tilde{\mathcal{Q}} \ni q \mapsto \tilde{\mathcal{J}}(q)$ is strictly convex. By convexity of $\tilde{\mathcal{S}}$, $\tilde{\mathcal{Q}}_{c_1}$ is also convex.

*Step 3: weak lower semi-continuity of $\tilde{\mathcal{S}}$.* We begin with further compactness properties of $\tilde{\mathcal{Q}}_{c_1}$. We thus consider a sequence $(q^k)_{k\in\mathbb{N}}$ lying in $\tilde{\mathcal{Q}}_{c_1}$. We denote by $(\tilde{Y}^{\star,k}, \tilde{Z}^{\star,k})_{k\in\mathbb{N}}$ the sequence of processes associated to $(q^k)_{k\in\mathbb{N}}$, as given by (87). We first prove that the two sequences $(\tilde{Y}^{\star,k})_{k\in\mathbb{N}}$ and $(\tilde{Z}^{\star,k})_{k\in\mathbb{N}}$ are relatively compact with respect to the weak topologies on $L^1(\Omega \times [0,T], \mathbb{R}, \mathbb{P} \otimes \mathrm{Leb}_{[0,T]})$ and $L^1(\Omega \times [0,T], \mathbb{R}^d, \mathbb{P} \otimes \mathrm{Leb}_{[0,T]})$ respectively. The relative compactness of $(\tilde{Y}^{\star,k})_{k\in\mathbb{N}}$ is established as in Step 1, by proving that the sequence is uniformly integrable. The latter is quite obvious: by (93), we know that the sequence $(|\tilde{Y}^{\star,k}|)_{k\in\mathbb{N}}$ is dominated by $(\alpha q^k)_{k\in\mathbb{N}}$; by Step 1, the sequence $(q^k)_{k\in\mathbb{N}}$ is uniformly integrable and, therefore, the sequence $(\tilde{Y}^{\star,k})_{k\in\mathbb{N}}$ is also uniform integrable. To prove the relative compactness of the sequence $(Z^{\star,k})_{k\in\mathbb{N}}$ in $L^1(\Omega\times[0,T],\mathbb{P}\otimes \mathrm{Leb}_{[0,T]},\mathbb{R}^d)$, we proceed as follows. By (103), we observe that, for any $k\in\mathbb{N}$, any subset $A\subset\Omega\times[0,T]$ in the progressive $\sigma$-field, and any real $\varepsilon>0$,

$$\begin{aligned}\mathbb{E}\left[\int_0^T \mathbb{1}_A(t)|\tilde{Z}_t^{k,\star}|\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^T \mathbb{1}_A(t)\mathbb{1}_{\{q_t^k>0\}}|\tilde{Z}_t^{k,\star}|\mathrm{d}t\right] \\ &\leq \frac{1}{\varepsilon}\mathbb{E}\left[\int_0^T \mathbb{1}_A(t)q_t^k\mathrm{d}t\right]+\varepsilon\mathbb{E}\left[\int_0^T \mathbb{1}_{\{q_t^k>0\}}\frac{1}{q_t^k}|\tilde{Z}_t^{k,\star}|^2\mathrm{d}t\right].\end{aligned}$$

Letting $Z_t^{k,\star}=\mathbb{1}_{\{q_t^k>0\}}\tilde{Z}^{k,\star}/q_t^k$, we deduce from (103) that there exists a constant $C$, independent of $k$, $A$ and $\varepsilon$, such that

$$\mathbb{E}\left[\int_0^T \mathbb{1}_A(t)|\tilde{Z}_t^{k,\star}|\mathrm{d}t\right]\leq \frac{1}{\varepsilon}\mathbb{E}\left[\int_0^T \mathbb{1}_A(t)q_t^k\mathrm{d}t\right]+C\varepsilon.$$

By the uniform integrability property established in the first step, we know that the first term on the right-hand side can be made as small as desired by choosing $\mathrm{Leb}_{[0,T]}\otimes\mathbb{P}(A)$ small enough. This proves that the collection $(\tilde{Z}^{\star,k})_{k\in\mathbb{N}}$ is uniformly integrable, on $\Omega\times[0,T]$, equipped with $\mathbb{P}\otimes\mathrm{Leb}_{[0,T]}$.

We now come back to the proof of the lower semi-continuity of $\tilde{\mathcal{S}}$. Let $(q^k)_{k\in\mathbb{N}}$ be a sequence in $\tilde{Q}_{c_1}$ that converges for the weak topology on $L^1(\Omega\times[0,T],\mathbb{P}\otimes\mathrm{Leb}_{[0,T]})$. The relative compactness of $(\tilde{Y}^{\star,k},\tilde{Z}^{\star,k})_{k\in\mathbb{N}}$ allows us to extract a subsequence (still indexed by $k$) that converges in $L^1(\Omega\times[0,T],\mathbb{R},\mathbb{P}\otimes\mathrm{Leb}_{[0,T]})\times L^1(\Omega\times[0,T],\mathbb{R}^d,\mathbb{P}\otimes\mathrm{Leb}_{[0,T]})$ equipped with the product of the weak topologies (on each factor). Since the objective is to prove the lower semi-continuity of the convex functional $\tilde{\mathcal{S}}$, we can replace $(q^k,\tilde{Y}^{\star,k},\tilde{Z}^{\star,k})_{k\in\mathbb{N}}$ by a convex combination that converges for the strong topologies. We denote by $(q,\tilde{Y}^\star,\tilde{Z}^\star)$ its (strong) limit. The objective is to pass to the limit in (87) and to prove that $q$ is a non-negative valued process that can be expanded as

$$\forall t\in[0,T],\quad q_t=1+\int_0^t \tilde{Y}_s^\star\mathrm{d}s+\int_0^t \tilde{Z}_s^\star\cdot\mathrm{d}W_s. \tag{105}$$

In fact, the difficulty is to pass to the limit in the stochastic integrals appearing in the expansion of each $q^k$. By (96),

$$\sup_{k\in\mathbb{N}}\mathbb{E}\left[\left(\int_0^T |\tilde{Z}_t^{\star,k}|^2\mathrm{d}t\right)^{1/2}\right]<+\infty.$$

By Fatou's lemma, we deduce that $\tilde{Z}^\star$ satisfies the same bound. And then, by a new uniform integrability argument (combining with the convergence in $L^1$),

$$\forall\eta\in(0,1),\quad \lim_{k\to\infty}\mathbb{E}\left[\left(\int_0^T |\tilde{Z}_t^{\star,k}-\tilde{Z}_t^\star|^2\mathrm{d}t\right)^{\eta/2}\right]=0,$$

which shows that

$$\forall\eta\in(0,1),\quad \lim_{k\to\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t\left(\tilde{Z}_s^{\star,k}-\tilde{Z}_s^\star\right)\cdot\mathrm{d}W_s\right|^\eta\right]=0.$$

This makes it possible to derive (105). Moreover, using Fatou's lemma, we deduce from the following three inequalities (with the last one following from (103))

$$\mathbb{E}[q_T^{k,*}] \leq \exp(\alpha T), \quad \mathbb{E}\left[\int_0^T \frac{|\tilde{Y}^{\star,k}|}{q_t^k}\mathrm{d}t\right] \leq T\alpha, \quad \mathbb{E}\left[\int_0^T \frac{|\tilde{Z}^{\star,k}|^2}{q_t^k}\mathrm{d}t\right] \leq C + 2\beta c_1,$$

that

$$\mathbb{E}[q_T^*] \leq \exp(\alpha T), \quad \mathbb{E}\left[\int_0^T \mathbb{1}_{\{q_t=0, \tilde{Y}_t^\star \neq 0\}}\mathrm{d}t\right] = 0, \quad \mathbb{E}\left[\int_0^T \mathbb{1}_{\{q_t=0, \tilde{Z}_t^\star \neq 0\}}\mathrm{d}t\right] = 0.$$

Letting $Y_t^\star := \mathbb{1}_{\{q_t>0\}}\tilde{Y}_t^\star/q_t$ and $Z_t^\star := \mathbb{1}_{\{q_t>0\}}\tilde{Z}_t^\star/q_t$, we deduce that (98) and (99) hold true (even though we do not have yet that $q \in \tilde{\mathcal{Q}}_{c_1}$ which prevents us from applying Lemma 18 at this stage of the proof). We also have

$$\mathbb{E}\left[\int_0^T q_t|Z_t^\star|^2\mathrm{d}t\right] < +\infty.$$

In particular, $|Z_t^\star|$ is (a.e.) finite when $q_t > 0$, and we can write, $\mathrm{Leb}_{[0,T]} \otimes \mathbb{P}$ almost everywhere, $\tilde{Y}_t^\star = q_t Y_t^\star$ and $\tilde{Z}_t^\star = q_t Z_t^\star$. By repeating the proof of the first claim in item *ii* of Lemma 18, we also have that $\mathbb{P}(\{\sup_{t\in[0,T]} q_t > 0\} \cap \{\tau < T\}) = 0$, where $\tau := \inf\{t \in [0,T];\ q_t = 0\}$.

We now come back to the definition of $\tilde{\mathcal{Q}}_{c_1}$. Using the fact that $q^k \in \tilde{\mathcal{Q}}_{c_1}$ and letting $\tau^k := \inf\{t \in [0,T];\ q_t^k = 0\}$, for each $k \in \mathbb{N}$, we deduce from (95) (together with the first claim in item *ii* of Lemma 18, which allows us to derive the third line below) that

$$\begin{aligned}
\tilde{\mathcal{S}}(q^k) &= \mathbb{E}\left[\int_0^{\tau^k} q_s^k f^\star(s, Y_s^{\star,k}, Z_s^{\star,k})\mathrm{d}s\right] \\
&\geq \mathbb{E}\left[\int_0^{\tau^k} q_s^k \left(Y_s Y_s^{\star,k} + Z_s \cdot Z_s^{\star,k} - f(s, Y_s, Z_s)\right)\mathrm{d}s\right] \\
&= \mathbb{E}\left[\int_0^{T} \left(Y_s \tilde{Y}_s^{\star,k} + Z_s \cdot \tilde{Z}_s^{\star,k} - q_s^k f(s, Y_s, Z_s)\right)\mathrm{d}s\right],
\end{aligned} \tag{106}$$

where $(Y, Z)$ is taken, for a certain $R > 0$, in the ball of center 0 and radius $R$ of the space $L^\infty(\mathbb{F}) \times L^\infty(\mathbb{F}, \mathbb{R}^d)$, i.e. for almost every $(\omega, s) \in \Omega \times [0,T]$,

$$|Y_s(\omega)| + |Z_s(\omega)| \leq R.$$

Letting $k$ tend to $+\infty$ in (106), we deduce that, for any $R > 0$,

$$c_1 \geq \sup_{(Y,Z)\in\mathcal{B}_R} \mathbb{E}\left[\int_0^T q_s \left(Y_s Y_s^\star + Z_s \cdot Z_s^\star - f(s, Y_s, Z_s)\right)\mathrm{d}s\right]. \tag{107}$$

This prompts us to define, for any $R > 0$, the mapping $f_R^\star \colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, by letting

$$f_R^\star(t, y^\star, z^\star) := \sup_{|y|+|z|\leq R} \left\{y^\star y + z^\star \cdot z - f(t, y, z)\right\}. \tag{108}$$

Let us remark that $(f_R^\star)_{R\in\mathbb{N}}$ is a non-decreasing sequence, converging pointwise to $f^\star$. By [15, Proposition 3.78], we have that

$$\begin{aligned}
&\sup_{(Y,Z)\in\mathcal{B}_R} \mathbb{E}\left[\int_0^T q_s\left(Y_sY_s^\star + Z_s\cdot Z_s^\star - f(s,Y_s,Z_s)\right)\mathrm{d}s\right] \\
&= \mathbb{E}\left[\int_0^T q_s f_R^\star(s,Y_s^\star,Z_s^\star)\mathrm{d}s\right] = \mathbb{E}\left[\int_0^\tau q_s f_R^\star(s,Y_s^\star,Z_s^\star)\mathrm{d}s\right],
\end{aligned} \tag{109}$$

where we recall that $\tau = \inf\{t\in[0,T];\ q_t = 0\}$. Combining (107) and (109) together with Fatou's lemma, we obtain

$$\begin{aligned}
c_1 &\geq \liminf_{R\to+\infty} \mathbb{E}\left[\int_0^\tau q_s f_R^\star(s,q_s,\tilde{Y}_s^\star,\tilde{Z}_s^\star)\mathrm{d}s\right] \\
&\geq \mathbb{E}\left[\int_0^\tau q_s f^\star(s,q_s,\tilde{Y}_s^\star,\tilde{Z}_s^\star)\mathrm{d}s\right] = \tilde{\mathcal{S}}(q),
\end{aligned} \tag{110}$$

with the last identity following from (95) (which holds true here, thanks to the analysis achieved in the first part of this step). This shows that $\tilde{\mathcal{S}}$ is weakly lower semi-continuous.

*Step 4: weak compactness of $\tilde{\mathcal{Q}}_{c_1}$.* Most of the work has been done in the previous steps. We know from the first step that any sequence $(q^k)_{k\in\mathbb{N}}$ is relatively compact for the weak topology on $L^1(\Omega\times[0,T],\mathbb{P}\otimes\mathrm{Leb}_{[0,T]})$. By (105) in Step 3, we know that any weak limit $q$ can be expanded as in (87), is non-negative valued and satisfies $\mathbb{E}[q_T^*]\leq\exp(\alpha T)$. And by (110), $\tilde{\mathcal{S}}(q)\leq c_1$.

*Step 5: weak continuity of $\mathcal{R}$.* We are thus given a sequence $(q^k)_{k\in\mathbb{N}}\in\tilde{\mathcal{Q}}_{c_1}$. By the previous step, there exists $q\in\tilde{\mathcal{Q}}_{c_1}$ such that, up to a subsequence, $(q^k)_{k\in\mathbb{N}}$ weakly converges (i.e., with respect to $\sigma(L^1,L^\infty)$) to $q$ as $k\to+\infty$. By concavity of $\mathcal{G}$, we have (with the shorthand notation $\mathcal{G}(q)$ for $\mathcal{G}(q,X_T^\psi)$, and similarly for the derivative $\delta_q\mathcal{G}(q_T)$)

$$\begin{aligned}
\mathcal{R}(q^k) &= \mathcal{G}\left(q_T^k\right) + \mathbb{E}\left[\int_0^T q_s^k\ell_s\mathrm{d}s\right] \\
&\leq \mathcal{G}\left(q_T\right) + \mathbb{E}\left[\left(q_T^k - q_T\right)\delta_q\mathcal{G}(q_T) + \int_0^T q_s^k\ell_s\mathrm{d}s\right] \\
&= \mathcal{R}\left(q\right) + \mathbb{E}\left[\left(q_T^k - q_T\right)\delta_q\mathcal{G}\left(q_T\right) + \int_0^T\left(q_s^k - q_s\right)\ell_s\mathrm{d}s\right].
\end{aligned} \tag{111}$$

Recalling the growth Assumption A6 on $\mathcal{G}$, we have

$$|\delta_q\mathcal{G}\left(q_T\right)| = \left|\delta_q\mathcal{G}\left(q_T,X_T^\psi\right)\right| \leq L\left(1+|X_T^\psi|^{2-r}+\mathbb{E}\left[q_T|X_T^\psi|^{2-r}\right]\right),$$

and then, $\delta_q\mathcal{G}$ admits exponential moments of all orders since $\psi$ is assumed to be bounded. Thanks to Step 1, Lemma 43 (applicable by Remark 21) yields that the sequences

$$(q_T^k\delta_q\mathcal{G}\left(q_T\right))_{k\in\mathbb{N}},\quad (q_s^k\ell_s)_{k\in\mathbb{N}}$$

are uniformly integrable. Since $(q^k)_{k\in\mathbb{N}}$ is assumed to converge weakly with respect to $\sigma(L^1,L^\infty)$, we deduce from Lemma 44 (which is applicable even though the sequence $(q^k)_{k\in\mathbb{N}}$ takes values in $\tilde{\mathcal{Q}}_{c_1}$ and possibly not in $\mathcal{Q}_{c_1}$) that

$$\lim_{k\to+\infty}\mathbb{E}\left[\left(q_T^k - q_T\right)\delta_q\mathcal{G}\left(q_T\right) + \int_0^T\left(q_s^k - q_s\right)\ell_s\mathrm{d}s\right] = 0.$$

Then $\limsup_{k\to\infty} \mathcal{R}(q^k) \le \mathcal{R}(q)$ concluding the step.

*Step 6: Conclusion.* The statement follows from the combination of Steps 1 to 6. $\square$

**Lemma 22.** *Let $c_1, c_2 > 0$ and $q \in \tilde{\mathcal{Q}}_{c_1}$. The mapping $\mathcal{A}_{c_2} \ni \psi \mapsto \tilde{\mathcal{J}}(q,\psi)$ is convex, lower semi-continuous. In addition, $\mathcal{A}_{c_2}$ is convex and weakly compact in $L^2(\mathbb{F}, \mathbb{Q}^0, \mathbb{R}^n)$, where $\mathbb{Q}^0 := \mathcal{E}_T(\int_0^\cdot \partial_z f(t,0,0)\cdot \mathrm{d}W_t)\mathbb{P}$. In particular, it is bounded in $M^\eta(\mathbb{F}, \mathbb{R}^n, \mathbb{P})$, for any $\eta \in (0,1)$.*

*Proof. Step 1: convexity and weak compactness of $\mathcal{A}_{c_2}$.* The convexity of $\mathcal{S}^\star$, regarded as a $[0,+\infty]$-valued mapping defined on the space of $\mathbb{F}$-progressively measurable $\mathbb{R}^n$-valued processes, is a direct consequence of its definition (6): for each $q \in \mathcal{Q}$, the mapping $\psi \mapsto \mathbb{E}[\int_0^T q_s|\psi_s|^2 \mathrm{d}s]$ is convex since $q$ takes non-negative values. Therefore, $\mathcal{S}^\star$ is convex as the supremum of a family of convex mappings. As a result, the set $\mathcal{A}_{c_2}$ is convex.

The relative weak compactness of $\mathcal{A}_{c_2}$ in $L^2(\mathbb{F}, \mathbb{Q}^0, \mathbb{R}^n)$ is a direct consequence of the fact that

$$c_2 \ge \mathcal{S}^\star(\psi) = \sup_{q\in\mathcal{Q}} \left\{ \mathbb{E}\left[\int_0^T q_t|\psi_t|^2 \mathrm{d}t\right] - \frac{1}{\gamma}\mathcal{S}(q) \right\} \ge \mathbb{E}\left[q_T^0 \int_0^T |\psi_t|^2 \mathrm{d}t\right] - \frac{1}{\gamma}\mathcal{S}(q^0),$$

with $q^0$ as in (84). Then, we are left to prove the weak closure property of $\mathcal{A}_{c_2}$. By convexity of the latter, it suffices to show that it is closed for the strong topology (on $L^2(\mathbb{F}, \mathbb{R}^n, \mathbb{Q}^0)$). We thus consider an $\mathcal{A}_{c_2}$-valued sequence $(\psi^k)_{k\in\mathbb{N}}$ that (strongly) converges in $L^2(\mathbb{F}, \mathbb{R}^n, \mathbb{Q}^0)$ to some limit $\bar{\psi}$. Then, for all $q \in \mathcal{Q}$,

$$\mathbb{E}\left[\int_0^T q_t|\psi_t^k|^2 \mathrm{d}t\right] - \frac{1}{\gamma}\mathcal{S}(q) \le c_2.$$

By Fatou's lemma, the mapping $\psi \mapsto \mathbb{E}[\int_0^T q_t|\psi_t|^2 \mathrm{d}t]$ is lower semi-continuous for the strong topology on $L^2(\mathbb{F}, \mathbb{R}^n, \mathbb{Q}^0)$. We deduce that

$$\mathbb{E}\left[\int_0^T q_t|\psi_t|^2 \mathrm{d}t\right] - \frac{1}{\gamma}\mathcal{S}(q) \le c_2,$$

which implies that $\bar{\psi} \in \mathcal{A}_{c_2}$, as required.

Observing that $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}^0$ has finite exponential moments of any order under $\mathbb{Q}^0$ and using again the fact that $\sup_{\psi\in\mathcal{A}_{c_2}} \mathbb{E}^{\mathbb{Q}^0}[\int_0^T |\psi_t|^2 \mathrm{d}t] < +\infty$, we deduce that $\mathcal{A}_{c_2}$ is a bounded subset of $M^\eta(\mathbb{F}, \mathbb{R}^n, \mathbb{P})$ for any $\eta \in (0,2)$.

*Step 2: convexity of $\mathcal{A}_{c_2} \ni \psi \mapsto \tilde{\mathcal{J}}(q,\psi)$, for a fixed $q \in \tilde{\mathcal{Q}}_{c_1}$.* The convexity of the mapping

$$\mathcal{A}_{c_2} \ni \psi \mapsto \mathbb{E}\left[\int_0^T q_s \ell(s,\psi_s)\mathrm{d}s\right],$$

is a direct consequence of the convexity of $\ell$ and the non-negativity of $q$. We now turn to the convexity of the mapping $\mathcal{A}_{c_2} \ni \psi \mapsto \mathcal{G}(q_T, X_T^\psi)$. For $\psi^0, \psi^1 \in \mathcal{A}_{c_2}$, we denote by $(X^i)_{i=0,1}$ the solutions to the two state equations associated with $\psi^0$ and $\psi^1$ respectively. Moreover, for any $\theta \in (0,1)$, we call $X^\theta$ the solution to the state equation associated with $\theta\psi^1 + (1-\theta)\psi^0$. We notice that

$$X_T^\theta = \theta X_T^1 + (1-\theta) X_T^0.$$

Using the fact that $\mathcal{G}$ is convex with respect to its second variable by Assumption A8 together with the last equality, it is easy to deduce that $[0,1] \ni \theta \mapsto \mathcal{G}(q_T, X_T^\theta)$ is convex. This concludes the step.

*Step 3: lower semi-continuity of $\mathcal{A}_{c_2} \ni \psi \mapsto \tilde{\mathcal{J}}(q,\psi)$, for a fixed $q \in \tilde{\mathcal{Q}}_{c_1}$.* The proof relies on the fact that, as explained in Remark 21, the duality inequality (14) extends to elements $q \in \tilde{\mathcal{Q}}_{c_1}$.

Let $(\psi^k)_{k\in\mathbb{N}}$ be a sequence with values in $\mathcal{A}_{c_2}$. By the extended version of (14), we have, for every $k \in \mathbb{N}$,

$$\gamma \mathbb{E}\left[\int_0^T q_t |\psi_t^k|^2 \mathrm{d}t\right] \leq \mathcal{S}^\star(\psi^k) + \tilde{\mathcal{S}}(q) \leq c_1 + c_2,$$

from which we deduce that the sequence $(\psi^k)_{k\in\mathbb{N}}$ lies in a weakly compact subset of the space of $\mathbb{F}$-progressively measurable $\mathbb{R}^n$-valued processes that are square integrable under the measure $\mathbb{Q}$, defined by $\mathbb{Q}(E) := \mathbb{E}\int_0^T \mathbb{1}_E q_t \mathrm{d}t$, for any event of $\Omega \times [0,T]$; with a slight abuse of notation, we will denote this space by $L^2(\mathbb{F},\mathbb{R}^n,\mathbb{Q})$. Up to a subsequence, there exists a weak limit in $L^2(\mathbb{F},\mathbb{R}^n,\mathbb{Q})$, which we denote $\bar{\psi}$. As the purpose is to prove that $\mathcal{J}(q,\bar{\psi}) \leq \liminf_{k\to+\infty} \mathcal{J}(q,\psi^k)$, and the functional $\mathcal{J}$ is convex with respect to the first argument, we can replace the sequence $(\psi^k)_{k\in\mathbb{N}}$ by a sequence of convex combinations (of the $(\psi_k)_{k\in\mathbb{N}}$'s) that converges in $L^2(\mathbb{F},\mathbb{R}^n,\mathbb{Q})$ (equipped with the strong topology) to $\bar{\psi}$. Following the second step in the proof of Lemma 18, we know that there exists a constant $c > 0$ such that $q_t \geq c\mathbb{E}[q_T|\mathcal{F}_t]$, for all $t \in [0,T]$. In particular, if $E$ is in the progressive $\sigma$-field, then

$$\mathbb{Q}(E) \geq c\mathbb{E}\left[q_T \mathbb{1}_E\right]. \tag{112}$$

By convexity of the function $\mathcal{G}$ in the second argument, see A8, we also have

$$\forall k \in \mathbb{N}, \quad \mathcal{J}(q,\psi^k) \geq \mathcal{J}(q,\bar{\psi}) + \mathbb{E}\left[A^k + B^k\right], \tag{113}$$

where

$$A^k := \delta_X \mathcal{G}\left(q_T, X_T^{\bar{\psi}}\right) \cdot \left(X_T^{\psi^k} - X_T^{\bar{\psi}}\right), \quad B^k := \int_0^T q_s \left(\ell(s,\psi_s^k) - \ell(s,\bar{\psi}_s)\right) \mathrm{d}s.$$

We study the two sequences of random variables $(A^k)_{k\in\mathbb{N}}$ and $(B^k)_{k\in\mathbb{N}}$ separately. We start with $(A^k)_{k\in\mathbb{N}}$. By the growth Assumption A6 on $\delta_X\mathcal{G}$, we have

$$|A^k| \leq L q_T \left(1 + |X_T^{\bar{\psi}}|^{1-r} + \mathbb{E}\left[q_T |X_T^{\bar{\psi}}|^{2-r}\right]\right) \left|X_T^{\psi^k} - X_T^{\bar{\psi}}\right|. \tag{114}$$

By Lemma 40 in Appendix B (together with Remark 21), we know that

$$\mathbb{E}\left[q_T |X_T^{\bar{\psi}}|^{2-r}\right] \leq C\left(1 + \tilde{\mathcal{S}}(q) + \mathcal{S}^\star(\bar{\psi})\right) \leq C,$$

since $\bar{\psi} \in L^2(\mathbb{F},\mathbb{R}^n,\mathbb{Q})$ and $q \in \tilde{\mathcal{Q}}_{c_1}$. Taking the expectation both sides of (114) yields

$$\mathbb{E}[|A^k|] \leq C\mathbb{E}\left[q_T\left(1 + \left|X_T^{\bar{\psi}}\right|^{1-r}\right)\left|X_T^{\psi^k} - X_T^{\bar{\psi}}\right|\right], \tag{115}$$

We distinguish between the cases $r = 0$ and $r = 1$.

*Sub-step 3a: analysis of* $(A^k)_{k\in\mathbb{N}}$ *when* $r=0$. When $r=0$, we have

$$\begin{aligned}\mathbb{E}[|A^k|] &\le L\mathbb{E}\left[q_T\left(1+\left|X_T^{\bar\psi}\right|\right)\left|X_T^{\psi^k}-X_T^{\bar\psi}\right|\right]\\ &\le C\mathbb{E}\left[q_T\left(1+\left|X_T^{\bar\psi}\right|\right)\int_0^T\left|\psi_t^k-\bar\psi_t\right|\mathrm{d}t\right]\\ &\le C\mathbb{E}\left[q_T\left(1+\left|X_T^{\bar\psi}\right|^2\right)\right]^{1/2}\mathbb{E}\left[q_T\int_0^T\left|\psi_t^k-\bar\psi_t\right|^2\mathrm{d}t\right]^{1/2},\end{aligned}$$

where the last line follows from the Cauchy-Schwarz inequality, for some constant $C>0$. Then, $\lim_{k\to+\infty}\mathbb{E}[|A^k|]=0$, by (112) and strong convergence of $(\psi^k)_{k\in\mathbb{N}}$ in $L^2(\mathbb{F},\mathbb{R}^n,\mathbb{Q})$.

*Sub-step 3b: analysis of* $(A^k)_{k\in\mathbb{N}}$ *when* $r=1$. When $r=1$, there exists a constant $C>0$ such that

$$\begin{aligned}\mathbb{E}[|A^k|] &\le C\mathbb{E}\left[q_T\left|X_T^{\psi^k}-X_T^{\bar\psi}\right|\right]\\ &\le C\left(\mathbb{E}\left[q_T\int_0^T\left|\psi_t^k-\bar\psi_t\right|\mathrm{d}t\right]+\mathbb{E}\left[q_T\left|\int_0^T\left(\sigma(t,\psi_t^k)-\sigma(t,\bar\psi_t)\right)\mathrm{d}W_t\right|\right]\right)\\ &=: a_1^k+a_2^k. \end{aligned}\tag{116}$$

Clearly, $(a_1^k)_{k\in\mathbb{N}}$ converges to 0 as $k\to+\infty$. To handle $(a_2^k)_{k\in\mathbb{N}}$, we use the same family $(q^\theta)_{\theta\in[0,1)}$ as in Remark 21. We recall that each $q^\theta$ is positive valued and belongs to $\mathcal{Q}_{c_1}$ (provided that $c_1$ is large enough, which is not a restriction here). Below, we write the expansion (4) of $q^\theta$, for $\theta\in[0,1)$, in the form

$$\mathrm{d}q_t^\theta = q_t^\theta Y_t^{\star,\theta}\mathrm{d}t + q_t^\theta Z_t^{\star,\theta}\cdot\mathrm{d}W_t,\quad t\in[0,T],$$

with initial condition $q_0^\theta=1$. We notice that

$$a_2^k \le \frac{1}{1-\theta}\mathbb{E}\left[q_T^\theta\left|\int_0^T\left(\sigma(t,\psi_t^k)-\sigma(t,\bar\psi_t)\right)\mathrm{d}W_t\right|\right] =: \frac{1}{1-\theta}a_2^{k,\theta}.$$

By (102), we have $\sup_{\theta\in[0,1]}\mathbb{E}[h(q_T^\theta)]<+\infty$. By Lemma 39, we also know that $\mathbb{Q}^\theta := \mathcal{E}_T(\int_0^\cdot Z_t^{\star,\theta}\cdot\mathrm{d}W_t)\mathbb{P}$ is a probability measure. It satisfies $\exp(-\alpha T)q_T^\theta \le \mathrm{d}\mathbb{Q}^\theta/\mathrm{d}\mathbb{P}\le\exp(\alpha T)q_T^\theta$. Therefore, letting $(\tilde W_t^\theta := W_t-\int_0^t Z_s^{\star,\theta}\mathrm{d}s)_{t\in[0,T]}$, we deduce from Girsanov's theorem that

$$\begin{aligned}a_2^{k,\theta} &\le C\mathbb{E}^{\mathbb{Q}^\theta}\left[\left|\int_0^T\left(\sigma(t,\psi_t^k)-\sigma\left(t,\bar\psi_t\right)\right)\mathrm{d}\tilde W_t^\theta\right|\right]\\ &\quad + C\mathbb{E}^{\mathbb{Q}^\theta}\left[\left|\int_0^T\left(\sigma(t,\psi_t^k)-\sigma\left(t,\bar\psi_t\right)\right)Z_t^{\star,\theta}\mathrm{d}t\right|\right]\\ &\le C\mathbb{E}^{\mathbb{Q}^\theta}\left[\left|\int_0^T\left|\psi_t^k-\bar\psi_t\right|^2\mathrm{d}t\right|\right]^{1/2}\left(1+\mathbb{E}^{\mathbb{Q}^\theta}\left[\int_0^T|Z_t^{\star,\theta}|^2\mathrm{d}t\right]^{1/2}\right).\end{aligned}\tag{117}$$

By Lemma 39 and thanks to the bound $\sup_{\theta\in[0,1)}\mathbb{E}[h(q_T^\theta)]<+\infty$, we have

$$\sup_{\theta\in[0,1)}\mathbb{E}^{\mathbb{Q}^\theta}\left[\int_0^T|Z_t^{\star,\theta}|^2\mathrm{d}t\right]<+\infty.\tag{118}$$

As for the first term on the last line of (117), we notice that

$$\mathbb{E}^{\mathbb{Q}^\theta}\left[\int_0^T \left|\psi_t^k - \bar{\psi}_t\right|^2 \mathrm{d}t\right] \le \exp(\alpha T)\mathbb{E}\left[q_T^\theta \int_0^T \left|\psi_t^k - \bar{\psi}_t\right|^2 \mathrm{d}t\right]$$
$$= (1-\theta)\exp(\alpha T)\mathbb{E}\left[q_T^0 \int_0^T \left|\psi_t^k - \bar{\psi}_t\right|^2 \mathrm{d}t\right] \tag{119}$$
$$+ \theta\mathbb{E}\left[q_T \int_0^T \left|\psi_t^k - \bar{\psi}_t\right|^2 \mathrm{d}t\right].$$

By Step 1, we know that the first term on the right-hand side is less than $C(1-\theta)$. Using the fact that $(\psi^k)_{k\in\mathbb{N}}$ strongly converges to $\bar{\psi}$ in $L^2(\mathbb{F}, \mathbb{R}^n, \mathbb{Q})$, the second one tends to 0 as $k$ tends to $+\infty$ (uniformly in $\theta \in [0,1)$). By (117), (118) and (119), we deduce that $\lim_{\theta\to 1}\lim_{k\to+\infty} a_2^{k,\theta} = 0$. By (116), we obtain $\lim_{k\to+\infty}\mathbb{E}[|A^k|] = 0$ when $r = 1$.

*Sub-step 3c: analysis of* $(B^k)_{k\in\mathbb{N}}$. Back to (113), we now study the sequence $(B^k)_{k\in\mathbb{N}}$. Using the fact that the gradient of $\ell$ in the second variable is at most of linear growth, see A4, and once again the fact that $(\psi^k)_{k\in\mathbb{N}}$ strongly converges to $\psi$ in $L^2(\mathbb{F}, \mathbb{R}^n, \mathbb{Q})$, we directly obtain

$$\lim_{k\to+\infty} B^k = \lim_{k\to+\infty}\left(\mathbb{E}\left[\int_0^T q_s \ell(s, \psi_s^k)\mathrm{d}s\right] - \mathbb{E}\left[\int_0^T q_s \ell\left(s, \bar{\psi}_s\right)\mathrm{d}s\right]\right) = 0.$$

*Sub-step 3d: conclusion.* From the last three sub-steps, we deduce that $\mathbb{E}[A^k + B^k]$ in the right-hand side of (113) tends to 0. We deduce that $\liminf_{k\to+\infty}\mathcal{J}(\psi^k) \ge \mathcal{J}(\bar{\psi})$, which concludes the step and the proof. □

**Lemma 23.** *There exists a solution to problem* ($\tilde{\mathrm{P}}$'), *i.e. there exists* $(\bar{\psi}, \bar{q}) \in \mathcal{A}_{c_2} \times \tilde{\mathcal{Q}}_{c_1}$ *such that*

$$\min_{\psi\in\mathcal{A}_{c_2}}\max_{q\in\tilde{\mathcal{Q}}_{c_1}}\tilde{\mathcal{J}}(q,\psi) = \max_{q\in\tilde{\mathcal{Q}}_{c_1}}\min_{\psi\in\mathcal{A}_{c_2}}\tilde{\mathcal{J}}(q,\psi) = \tilde{\mathcal{J}}(\bar{q},\bar{\psi}). \tag{120}$$

*Proof.* For each $k \in \mathbb{N}$, we call $\mathcal{B}_k^\infty$ the collection of $\mathbb{F}$-progressively measurable $\mathbb{R}^n$-valued processes that are bounded by $k$ on a subset of full measure under $\mathbb{P}\otimes\mathrm{Leb}_{[0,T]}$. We then view $\mathcal{A}_{c_2} \cap \mathcal{B}_k^\infty$ as a subset of $L^2(\mathbb{F}, \mathbb{R}^n, \mathbb{Q}^0)$, with $\mathbb{Q}^0$ being defined as in the statement of Lemma 22. By Lemma 22 again, $\mathcal{A}_{c_2} \cap \mathcal{B}_k^\infty$ is a convex and weakly compact subset of $L^2(\mathbb{F}, \mathbb{R}^n, \mathbb{Q}^0)$. By Theorem 1, which we can apply thanks to Proposition 20 and Lemma 22, we can find, for each $k \in \mathbb{N}$, a saddle point $(q^k, \psi^k) \in \tilde{\mathcal{Q}}_{c_1} \times (\mathcal{A}_{c_2} \cap \mathcal{B}_k^\infty)$ to the min-max problem

$$\min_{\psi\in\mathcal{A}_{c_2}\cap\mathcal{B}_k^\infty}\max_{q\in\tilde{\mathcal{Q}}_{c_1}}\tilde{\mathcal{J}}(q,\psi) = \max_{q\in\tilde{\mathcal{Q}}_{c_1}}\min_{\psi\in\mathcal{A}_{c_2}\cap\mathcal{B}_k^\infty}\tilde{\mathcal{J}}(q,\psi) = \tilde{\mathcal{J}}(q^k,\psi^k). \tag{121}$$

By compactness of $\tilde{\mathcal{Q}}_{c_1}$ and $\mathcal{A}_{c_2}$, for the weak topologies $\sigma(L^1(\mathbb{F},\mathbb{R},\mathbb{P}), L^\infty(\mathbb{F},\mathbb{R},\mathbb{P}))$ and $\sigma(L^2(\mathbb{F},\mathbb{R}^n,\mathbb{Q}^0), L^2(\mathbb{F},\mathbb{R}^n,\mathbb{Q}^0))$, the sequence $(q^k,\psi^k)_{k\in\mathbb{N}}$ converges, up to a subsequence, for the product topology. The limit is denoted $(\bar{q},\bar{\psi})$. The objective of the proof is thus to show that

$$\tilde{\mathcal{J}}(q,\bar{\psi}) \le \tilde{\mathcal{J}}(\bar{q},\bar{\psi}) \le \tilde{\mathcal{J}}(\bar{q},\psi), \tag{122}$$

for all $\psi \in \mathcal{A}_{c_2}$ and $q \in \tilde{\mathcal{Q}}_{c_1}$, which is known to be equivalent to the equality (120).

*Step 1: $\tilde{\mathcal{J}}(\bar{q},\bar{\psi}) \le \tilde{\mathcal{J}}(\bar{q},\psi)$ for any $\psi \in \mathcal{A}_{c_2}$.* For $0 \le k_0 \le k$ and $\psi \in \mathcal{A}_{c_2}$, let $(\psi_t^{k_0} := \psi_t \mathbb{1}_{\{|\psi_t| \le k_0\}})_{t\in[0,T]}$. By (121), we have that

$$\tilde{\mathcal{J}}(\bar{q},\psi^k) \le \tilde{\mathcal{J}}(q^k,\psi^k) \le \tilde{\mathcal{J}}(q^k,\psi^{k_0}).$$

Taking infimum and supremum limits in the last inequality, we have

$$\liminf_{k\to+\infty} \tilde{\mathcal{J}}(\bar{q},\psi^k) \le \limsup_{k\to+\infty} \tilde{\mathcal{J}}(\bar{q},\psi^k) \le \limsup_{k\to+\infty} \tilde{\mathcal{J}}(q^k,\psi^{k_0}). \tag{123}$$

We first handle the term on the right-hand side. By concavity of $\mathcal{G}$ in the first variable, we have

$$\tilde{\mathcal{J}}(q^k,\psi^{k_0}) \le \tilde{\mathcal{J}}(\bar{q},\psi^{k_0}) + a^k + b^k, \tag{124}$$

where

$$a^k := \mathbb{E}\left[\delta_q\mathcal{G}\left(\bar{q}_T, X_T^{\psi^{k_0}}\right)(q_T^k - \bar{q}_T) + \int_0^T (q_t^k - \bar{q}_t)\ell(t,\psi_t^{k_0})\mathrm{d}t\right], \quad b^k := \tilde{\mathcal{S}}(\bar{q}) - \tilde{\mathcal{S}}(q^k).$$

By boundedness of $\psi^{k_0}$, the integrand $\ell(t,\psi_t^{k_0})$ is bounded, uniformly in $t$ and $\omega$. Moreover, $|X_T^{\psi^{k_0}}|^{2-r}$ has exponential moments of any order, which implies, from Lemma 40 and Remark 21, that $\mathbb{E}[q_T|X_T^{\psi^{k_0}}|^{2-r}] < +\infty$. And then, by the growth Assumption A6 on $\delta_q\mathcal{G}$, we deduce that that $\delta_q\mathcal{G}(q_T, X_T^{\psi^{k_0}})$ has exponential moments of any order. Recalling from (102) that

$$\sup_{t\in[0,T]} \mathbb{E}\left[h(\bar{q}_t)\right] < +\infty, \quad \sup_{k\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\left[h(q_t^k)\right] < +\infty,$$

Lemma 43 yields

$$\lim_{k\to+\infty} a^k = 0.$$

By weak lower semi-continuity of $\tilde{\mathcal{S}}$ on $\tilde{\mathcal{Q}}_{c_1}$ (see Step 3 in the proof of Proposition 20), we also have

$$\limsup_{k\to+\infty} b^k \le 0.$$

Therefore, inserting the last two displays in (124) and then returning back to (123), we obtain

$$\liminf_{k\to+\infty} \tilde{\mathcal{J}}(\bar{q},\psi^k) \le \tilde{\mathcal{J}}(\bar{q},\psi^{k_0}).$$

By weak lower semi-continuity of $\mathcal{A}_{c_2} \ni \psi \mapsto \tilde{\mathcal{J}}(\bar{q},\psi)$ (see Lemma 22, using the fact that $\bar{q} \in \mathcal{Q}_{c_1}$), the last inequality yields

$$\tilde{\mathcal{J}}(\bar{q},\bar{\psi}) \le \tilde{\mathcal{J}}(\bar{q},\psi^{k_0}). \tag{125}$$

It remains to pass to the limit in the right-hand side. By regularity of $\mathcal{G}$, we can write

$$\tilde{\mathcal{J}}(\bar{q},\psi^{k_0}) = \tilde{\mathcal{J}}(\bar{q},\psi) + \mathbb{E}\left[A^{k_0} + B^{k_0}\right], \tag{126}$$

where

$$\begin{aligned} A^{k_0} &:= \int_0^1 \delta_X\mathcal{G}(q_T, X_T^{\lambda,\psi^{k_0}}) \cdot (X_T^{\psi^{k_0}} - X_T^{\psi})\mathrm{d}\lambda, \\ B^{k_0} &:= \int_0^T \bar{q}_s(\ell(s,\psi_s^{k_0}) - \ell(s,\psi_s))\mathrm{d}s, \end{aligned} \tag{127}$$

and $X_T^{\lambda,\psi^{k_0}} := \lambda X_T^{\psi^{k_0}} + (1-\lambda)X_T^{\psi}$. At this point, we are in a situation very similar to (113), except for the fact that $\delta_X \mathcal{G}$ in the definition of $A^{k_0}$ is computed at $X^{\lambda,\psi^{k_0}}$. Apart from this, the context is the same. In particular, with the same abuse of notation as in the third step of the proof of Lemma 22, $\psi^{k_0}$ converges to $\psi$ (as $k_0$ tends to $+\infty$) in $L^2(\mathbb{F}, \mathbb{R}^n, \bar{\mathbb{Q}})$, where $\bar{\mathbb{Q}}$ is defined by $\bar{\mathbb{Q}}(E) = \mathbb{E}\int_0^T \mathbb{1}_E(t)\bar{q}_t \mathrm{d}t$. Indeed, the extended version of (14) (for elements $q$ in $\tilde{\mathcal{Q}}_{c_1}$, see again Remark 21) yields

$$\mathbb{E}\left[\bar{q}_T \int_0^T |\psi_t|^2 \mathrm{d}t\right] < +\infty.$$

Recalling that $(\psi_t^{k_0} := \psi \mathbb{1}_{\{|\psi_t| \le k_0\}})_{t\in[0,T]}$, we deduce from dominated convergence theorem that $\psi^{k_0}$ indeed converges to $\psi$ (as $k_0 \to +\infty$) in $L^2(\mathbb{F}, \mathbb{R}^n, \bar{\mathbb{Q}})$. Following the exact same reasoning as in Step 3 of Lemma 22, we then obtain that

$$\lim_{k_0\to+\infty} \mathbb{E}\left[A^{k_0} + B^{k_0}\right] = 0. \tag{128}$$

Combining the last result with (125) and (126), we deduce that $\tilde{\mathcal{J}}(q,\bar{\psi}) \le \tilde{\mathcal{J}}(\bar{q},\bar{\psi})$, which completes the first step.

*Step 2:* $\tilde{\mathcal{J}}(q,\bar{\psi}) \le \tilde{\mathcal{J}}(\bar{q},\bar{\psi})$ *for any* $q \in \tilde{\mathcal{Q}}_{c_1}$. Because the proof follows arguments that are similar to those in the first step, we just give a sketch of it. For integers $0 \le k_0 \le k$, we deduce, again from (121), that

$$\tilde{\mathcal{J}}(q,\psi^k) \le \tilde{\mathcal{J}}(q^k,\psi^k) \le \tilde{\mathcal{J}}(q^k,\bar{\psi}^{k_0}),$$

for any $q \in \tilde{\mathcal{Q}}_{c_1}$ and where $\bar{\psi}^{k_0} := \bar{\psi}\mathbb{1}_{\{|\bar{\psi}|\le k_0\}}$. By definition of the infimum and supremum limits, we have

$$\liminf_{k\to+\infty} \tilde{\mathcal{J}}(q,\psi^k) \le \limsup_{k\to+\infty} \tilde{\mathcal{J}}(q^k,\bar{\psi}^{k_0}),$$

Following the same arguments as in Step 1, we have that

$$\limsup_{k\to+\infty} \tilde{\mathcal{J}}(q^k,\bar{\psi}^{k_0}) = \tilde{\mathcal{J}}(\bar{q},\bar{\psi}^{k_0}), \quad \liminf_{k\to+\infty} \tilde{\mathcal{J}}(q,\psi^k) \ge \tilde{\mathcal{J}}(q,\bar{\psi}).$$

Combining the last inequalities yields

$$\tilde{\mathcal{J}}(q,\bar{\psi}) \le \tilde{\mathcal{J}}(\bar{q},\bar{\psi}^{k_0}).$$

Finally, using (126) in the right-hand side and then taking the limit $k_0 \to +\infty$, the conclusion of the step and the proof follows by (128). □

## 5.2 Nature's control problem

Given two constants $c_1, c_2 > 0$, we address the *restricted* Nature control problem

$$\sup_{q'\in\mathcal{Q}_{c_1}} \mathcal{J}(\bar{\psi}, q'), \tag{$\mathrm{P}_{\mathrm{N},c_1}$}$$

under the assumption that, for some $\bar{q}$, the pair $(\bar{q},\bar{\psi}) \in \mathcal{Q}_{c_1} \times \mathcal{A}_{c_2}$ is a saddle point of the problem (P'). In particular, the supremum in ($\mathrm{P}_{\mathrm{N},c_1}$) is equal to $\mathcal{J}(\bar{q},\bar{\psi})$ and our goal becomes to characterize $\bar{q}$ when $\bar{\psi}$ is given.

As a corollary of our analysis, we show that, when $c_1$ is sufficiently large, $\bar{q}$ is in fact the unique minimizer of the *unrestricted* Nature control problem

$$\sup_{q' \in \mathcal{Q}} \mathcal{J}(q', \bar{\psi}), \tag{P$_\mathrm{N}$}$$

which, in contrast with ($\mathrm{P}_{\mathrm{N},c_1}$), is set over the entire set $\mathcal{Q}$.

A key step in relaxing the constraint imposed on $q'$ in ($\mathrm{P}_{\mathrm{N},c_1}$), and thereby passing from $\mathcal{Q}_{c_1}$ to $\mathcal{Q}$, is to show that the component $\bar{q}$ of any saddle point $(\bar{q}, \bar{\psi})$ of (P') actually lies in the interior of $\mathcal{Q}_{c_1}$, provided that $c_1$ is sufficiently large. This is the content of the following result, proved in Subsection 5.2.1.

**Proposition 24.** *There exists a constant $c_1' > 0$, only depending on the data, such that, for any $c_1 > c_1'$ and any saddle point $(q, \psi)$ to (P') over $\mathcal{Q}_{c_1} \times \mathcal{A}_{c_2}$, the component $q$ of the saddle point necessarily belongs to $\mathcal{Q}_{c_1'}$.*

This a priori bound then allows us to apply perturbative arguments to characterize the solutions of ($\mathrm{P}_{\mathrm{N},c_1}$), and subsequently of ($\mathrm{P}_\mathrm{N}$), by means of a stochastic maximum principle. The results are summarized in the main statement below:

**Theorem 25.** *Let $\bar{\psi} \in \mathcal{A}_{c_2}$ and assume $c_1 > c_1'$ with $c_1'$ as in the statement of Proposition 24.*

1. *If, for some $c_0 \in (0, c_1)$, there exists $q \in \mathcal{Q}_{c_0}$ that solves ($\mathrm{P}_{\mathrm{N},c_1}$) (i.e., that maximizes $q' \mapsto \mathcal{J}(q', \bar{\psi})$ over $\mathcal{Q}_{c_1}$), then there exists a pair $(Y, Z)$ such that $(q, Y, Z)$ belongs to the set $\mathscr{Q}$ defined in (26), and solves the FBSDE ($\mathrm{Opt}_\mathrm{N}$) (with $\psi = \bar{\psi}$ therein).*

2. *Conversely, assume that there exists a solution $(\bar{q}, \bar{Y}, \bar{Z}) \in \mathscr{Q}$ to the FBSDE ($\mathrm{Opt}_\mathrm{N}$), then $(\bar{q}, \bar{\psi})$ is the unique optimizer of Nature's control problem ($\mathrm{P}_\mathrm{N}$).*

3. *In particular, if, for some $\bar{q} \in \mathcal{Q}_{c_1}$, $(\bar{q}, \bar{\psi})$ is a saddle point of the problem (P'), then $\bar{q}$ is the unique solution to ($\mathrm{P}_\mathrm{N}$) (i.e., is the unique maximizer of $q' \mapsto \mathcal{J}(q', \bar{\psi})$ over the entire $\mathcal{Q}$).*

The proof is based on a series of lemmas, which are proved in the next paragraph.

*Sketch of the proof of Theorem 25.* Taking for granted the statement of Proposition 24 and the results proven in the forthcoming Subsubsections 5.2.3 and 5.2.2, Theorem 25 can be established as follows.

The necessary condition is addressed in Subsubsection 5.2.2. We prove in Lemma 28 that, for any optimizer $q$ of ($\mathrm{P}_{\mathrm{N},c_1}$) that belongs to $\mathcal{Q}_{c_0}$ for some $c_0 \in (0, c_1)$, the BSDE (24) has a solution $(Y, Z) \in D(\mathbb{F}, \mathbb{Q}) \times (\cap_{\beta(0,1)} M^\beta(\mathbb{F}, \mathbb{R}^d, \mathbb{Q}))$ that satisfies the optimality condition in the last line in ($\mathrm{Opt}_\mathrm{N}$). This shows that $(q, Y, Z)$ belongs to $\mathscr{Q}$ and solves ($\mathrm{Opt}_\mathrm{N}$), and this proves the first assertion in the statement of Theorem 25.

The second assertion (i.e., the converse) is a direct consequence of Lemma 30.

It remains to establish the third assertion. Given $\bar{q} \in \mathcal{Q}_{c_1}$ such that $(\bar{q}, \bar{\psi})$ is a saddle point of (P'), we know from Proposition 24 that $\bar{q} \in \mathcal{Q}_{c_1'}$ for some $c_1' \in (0, c_1)$. By the first assertion in the statement of Theorem 25, we deduce that, there exists a pair $(Y, Z)$ such that $(\bar{q}, Y, Z)$ solves ($\mathrm{Opt}_\mathrm{N}$). By the second assertion, we deduce that $\bar{q}$ is the unique maximizer of $q' \mapsto \mathcal{J}(q', \bar{\psi})$ over the entire set $\mathcal{Q}$. □

Throughout the subsection, the parameters $c_1$ and $c_2$ appearing in ($\mathrm{P}_{\mathrm{N},c_1}$) are fixed.

### 5.2.1 A priori estimate

The purpose of this subsubsection is to establish the following a priori estimate, from which Proposition 24 follows as a direct consequence:

**Lemma 26.** *There exist two constants $c_1' > 0$ and $C \geq 0$, only depending on the data and independent of $c_1, c_2$, such that, for any $(q, \psi) \in \mathcal{Q}_{c_1} \times \mathcal{A}_{c_2}$, with $c_1 > c_1'$, satisfying*

$$\mathcal{J}(q, 0) \geq \mathcal{J}(q, \psi) \geq \mathcal{J}(q^0, \psi), \tag{129}$$

*where $q^0$ solves (84), it holds $q \in \mathcal{Q}_{c_1'}$ and $\sup_{t\in[0,T]} \mathbb{E}\left[h(q_t)\right] \leq C$.*

Assuming that $c_1$ satisfies (83), $q^0$ in the statement belongs to $\mathcal{Q}_{c_1}$. Notice also that the condition (129) is stated for an arbitrary pair $(q, \psi) \in \mathcal{Q}_{c_1} \times \mathcal{A}_{c_2}$, but is automatically satisfied by the saddle point $(\bar{q}, \bar{\psi})$ introduced in the beginning of Subsection 5.3, see ($\mathrm{P}_{\mathrm{N},c_1}$).

*Proof. Step 1.* We first establish a bound for $\mathcal{S}(q)$ in terms of $\mathbb{E}[h(q_T)]$. Recalling that $\mathcal{J}(q, \psi) = \mathcal{R}(q, \psi) - \mathcal{S}(q)$, by inequality (129) we have

$$\begin{aligned}\mathcal{S}(q) &= -\mathcal{J}(q, 0) + \mathcal{R}(q, 0) \leq -\mathcal{J}(q^0, \psi) + \mathcal{R}(q, 0) \\ &= -\mathcal{J}(q^0, \psi) + \mathcal{G}(X_T^0, q_T) + \mathbb{E}\left[\int_0^T q_s \ell(s, 0) \mathrm{d}s\right].\end{aligned} \tag{130}$$

We first provide a lower bound for $\mathcal{J}(q^0, \psi)$. Recalling (84), we have

$$\mathcal{J}\left(q^0, \psi\right) = \mathcal{G}(q_T^0, X_T^\psi) + \mathbb{E}\left[\int_0^T q_t^0 \ell(s, \psi_s) \mathrm{d}s\right] + \mathbb{E}\left[\int_0^T q_s^0 f(s, 0, 0) \mathrm{d}s\right].$$

Using the convexity of $\mathcal{G}$ in the variable $X$ (see (A6)) and the $L^{-1}$-strong convexity of $\ell$ in the variable $\psi$ (see A4), we deduce that there exists a constant $C$, independent of $\psi$, such that

$$\begin{aligned}&\mathcal{J}\left(q^0, \psi\right) \\ &\geq \mathcal{G}(q^0, 0) - C\left(1 + \mathbb{E}\left[q_T^0 \left|X_T^\psi\right|\right]\right) + \frac{1}{2L}\mathbb{E}\left[\int_0^T q_s^0 |\psi_s|^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^T q_s^0 f(s, 0, 0) \mathrm{d}s\right].\end{aligned}$$

Using the assumptions A1 and A2, and rewriting the dynamics of $X^\psi$ under the equivalent probability measure $\mathcal{E}_T(\int_0^\cdot \partial_z f(t, 0, 0) \cdot \mathrm{d}W_t)\mathbb{P}$, we have, for a new value of $C$,

$$\mathbb{E}\left[q_T^0 \left|X_T^\psi\right|\right] \leq C\left(1 + \mathbb{E}\left[\int_0^T q_s^0 |\psi_s|^2 \mathrm{d}s\right]^{1/2}\right).$$

Then, by combining the last two displays, there exists (a new) constant $C > 0$, only depending on the data and independent of $c_1, c_2$ and $q$, such that

$$\mathcal{J}\left(q^0, \psi\right) \geq -C. \tag{131}$$

Back to (130), we now make use of the duality inequality (13). By the latter, together with the growth Assumption A6 and the bound (131), we get

$$\begin{aligned}\mathcal{S}(q) &\leq -\mathcal{J}(q^0, \psi) + L\left(1 + \mathbb{E}\left[(1 + q_T)|X_T^0|^{2-r}\right]\right) + \mathbb{E}\left[\int_0^T q_s \ell(s, 0) \mathrm{d}s\right] \\ &\leq C + \frac{1}{\vartheta}\mathbb{E}\left[h(q_T)\right] + \mathbb{E}\left[\exp\left(\vartheta\left(\xi + \int_0^T \ell_s^0 \mathrm{d}s\right)\right)\right],\end{aligned} \tag{132}$$

where $\xi \coloneqq L|X_T^0|^{2-r}$, $\ell_s^0 \coloneqq \ell(s,0)$ and $\vartheta > \upsilon \coloneqq \beta e^{\alpha T}$. We recall that $\alpha$ and $\beta$ are given by (A3). Moreover, using the lower bound (16) for the dual driver, we have (similar to (103))

$$\mathcal{S}(q) \geq -\mathbb{E}\left[\int_0^T q_s|f_s^0|\mathrm{d}s\right] + \frac{1}{2\beta}\mathbb{E}\left[\int_0^T q_s|Z_s^\star|^2\mathrm{d}s\right]. \tag{133}$$

Combining the last two inequalities (132) and (133), we obtain

$$\frac{1}{2\beta}\mathbb{E}\left[\int_0^T q_s|Z_s^\star|^2\mathrm{d}s\right] \leq C_1 + \frac{1}{\vartheta}\mathbb{E}\left[h(q_T)\right] + \mathbb{E}\left[\exp\left(\vartheta\left(\xi + \int_0^T \ell_s^0\mathrm{d}s\right)\right)\right], \tag{134}$$

where $C_1 > 0$ is a finite constant, defined by $C_1 \coloneqq C + T\|f^0\|_{L^\infty(\mathbb{F})}$.

*Step 2.* We now provide another bound for the entropy which will lead us to the expected result when combined with the conclusion of the first step. To do so, we let $(\tilde{q}_t \coloneqq q_t \exp(-\alpha t))_{t\in[0,T]}$. Similar to (104), we have, by Itô's formula and for any stopping time $\tau$ such that $\int_0^\tau q_s|Z_s^\star|^2\mathrm{d}s$ and $\sup_{t\in[0,\tau]}|q_t|$ belong to $L^\infty(\mathcal{F}_T)$,

$$h(\tilde{q}_\tau) = -1 + \int_0^\tau \tilde{q}_s\ln(\tilde{q}_s)(Y_s^\star - \alpha)\mathrm{d}s + \frac{1}{2}\int_0^\tau \tilde{q}_s|Z_s^\star|^2\mathrm{d}s + H_T^\tau, \tag{135}$$

where $(H_t^\tau \coloneqq \int_0^{t\wedge\tau}\tilde{q}_s\ln(\tilde{q}_s)Z_s^\star\cdot\mathrm{d}W_s)_{t\in[0,T]}$. Since $(\sqrt{\tilde{q}_t}\ln(\tilde{q}_t))_{0\leq t\leq\tau}$ belongs to $L^\infty(\mathbb{F})$, $H^\tau$ is a square integrable martingale. By boundedness of the partial derivative of the driver with respect to its first variable, i.e. $|Y_s^\star| \leq \alpha$, we have that

$$\int_0^T \tilde{q}_s\ln(\tilde{q}_s)(Y_s^\star - \alpha)\mathrm{d}s \leq \int_0^T \tilde{q}_s\ln(\tilde{q}_s)(Y_s^\star - \alpha)\mathbb{1}_{\{\tilde{q}_s<1\}}\mathrm{d}s \leq 2\alpha e^{-1}T.$$

Returning to (135), inserting the above inequality and taking expectation, we get

$$\frac{1}{\beta}\mathbb{E}\left[h(\tilde{q}_\tau)\right] \leq C_2 + \frac{1}{2\beta}\mathbb{E}\left[\int_0^\tau \tilde{q}_s|Z_s^\star|^2\mathrm{d}s\right], \tag{136}$$

where $C_2 = 2\alpha\beta^{-1}e^{-1}T - 1$. By a standard localization argument, we choose $\tau$ along a non-decreasing sequence of stopping times $(\tau_k)_{k\in\mathbb{N}}$ converging to $T$, such that $\int_0^{\tau_k} q_s|Z_s^\star|^2\mathrm{d}s$ and $\sup_{t\in[0,\tau_k]}|q_t|$ belong to $L^\infty(\mathcal{F}_T)$ for each $k\in\mathbb{N}$. This is possible to construct such a sequence because, by finiteness of $\mathbb{E}[h(q_T)]$, we have

$$\mathbb{E}\left[\int_0^T q_s|Z_s^\star|^2\mathrm{d}s\right] < +\infty, \quad \text{and} \quad \mathbb{E}\left[q_T^*\right] < +\infty,$$

with the second inequality following from $L\log L$-Doob's inequality. Observing that $h(\tilde{q}_\tau)$ is lower bounded by $-1/e - e$, we deduce from (136) and Fatou's lemma that

$$\frac{1}{\beta}\mathbb{E}\left[h(\tilde{q}_T)\right] \leq C_2 + \frac{1}{2\beta}\mathbb{E}\left[\int_0^T \tilde{q}_s|Z_s^\star|^2\mathrm{d}s\right]. \tag{137}$$

Moreover, because $\tilde{q}_T = \exp(-\alpha T)q_T$,

$$h(\tilde{q}_T) = \exp(-\alpha T)h(q_T) - \alpha T\exp(-\alpha T)q_T. \tag{138}$$

By the duality inequality (13), we have for any $\theta > 1$,

$$\alpha T\exp(-\alpha T)q_T \leq \frac{1}{\theta}\exp(-\alpha T)h(q_T) + \exp(\theta\alpha T). \tag{139}$$

Then, combining (138) and (139) yields

$$\left(1-\frac{1}{\theta}\right)\exp(-\alpha T)h(q_T)\leq h(\tilde{q}_T)+\exp(\theta\alpha T).$$

Combining the last inequality with (137), we obtain

$$\left(1-\frac{1}{\theta}\right)\frac{1}{\upsilon}\mathbb{E}\left[h(q_T)\right]\leq C_2+\frac{1}{2\beta}\mathbb{E}\left[\int_0^T\tilde{q}_s|Z_s^\star|^2\mathrm{d}s\right]+\exp(\theta\alpha T) \tag{140}$$

$$\leq C_2+\frac{1}{2\beta}\mathbb{E}\left[\int_0^T q_s|Z_s^\star|^2\mathrm{d}s\right]+\exp(\theta\alpha T), \tag{141}$$

where we recall that $\upsilon=\beta\exp(\alpha T)$, see (132).

*Step 3: Conclusion.* Combining (134) and (140), we get in the end

$$\left(\left(1-\frac{1}{\theta}\right)\frac{1}{\upsilon}-\frac{1}{\vartheta}\right)h(q_T)\leq C+\exp(\theta\alpha T)+\mathbb{E}\left[\exp\left(\vartheta\left(\xi+\int_0^T\ell_s^0\mathrm{d}s\right)\right)\right],$$

where $C=C_1+C_2$. Choosing $\theta>(1-\upsilon/\vartheta)^{-1}$, recalling that $\vartheta>\upsilon$ by definition (see again (132)), we deduce that there exists a finite constant $c_1'>0$ independent of $c_1$ such that $\sup_{t\in[0,T]}\mathbb{E}[h(q_t)]\leq c_1'$ provided that $\xi+\int_0^T\ell_s^0\mathrm{d}s\in L^{1,\vartheta}_{\exp}(\mathcal{F}_T)$. The latter can be verified by combining Lemma 42 with assumptions A4 and A5. The argument was already outlined in Remark 7. On the one hand, A4 says that $\ell^0$ is bounded by $L$. In particular, it suffices to show that $\xi$, which is here equal to $L|X_T^0|^{2-r}$, belongs to $L^{1,\vartheta}_{\exp}(\mathcal{F}_T)$, for some $\vartheta>\upsilon$. By Lemma 42, this is always true when $r=1$. When $r=0$, we need $4\vartheta L\|\Gamma\|^2_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\Gamma^{-1}\|^2_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\nu\|^2_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})}T<1$. By A5, this is indeed possible to choose $\vartheta$ satisfying the latter while ensuring the condition $\vartheta>\upsilon$. This completes the proof.

Finally, using (132) one last time, we conclude that $\mathcal{S}(q)\leq c_1'$, for a possibly new (but still independent of $c_1$) value of $c_1'$. □

### 5.2.2 Necessary condition

We show the necessary condition, i.e. the first assertion, in the statement of Theorem 25. Because $\bar{\psi}\in\mathcal{A}_{c_2}$ is fixed throughout the subsubsection, we omit it in most of the notations and merely write

$$\mathcal{J}(q):=\mathcal{J}(q,\bar{\psi}),\quad \mathcal{R}(q):=\mathcal{R}(q,\bar{\psi}),\quad \ell_s:=\ell(s,\bar{\psi}_s), \tag{142}$$

for $q\in\mathcal{Q}_{c_1}$. We denote by $(Y^\star,Z^\star)$ the representatives of $q$, as defined in (4), i.e. $(Y^\star,Z^\star)$ is an $\mathbb{R}\times\mathbb{R}^d$-valued $\mathbb{F}$-progressively measurable pair satisfying $\mathcal{S}(q)<c_1$. With $q$, we also associate the equivalent probability measure $\mathbb{Q}$ given by $\mathrm{d}\mathbb{Q}=\mathcal{E}_T(\int_0^\cdot Z_s^\star\cdot\mathrm{d}W_s)\mathrm{d}\mathbb{P}$. We also consider the adjoint BSDE, with unknown $(Y,Z)$,

$$\begin{cases} -\mathrm{d}Y_t &= (Y_t^\star Y_t+Z_t^\star\cdot Z_t-f^\star(t,Y_t^\star,Z_t^\star)+\ell_t)\,\mathrm{d}t-Z_t\cdot\mathrm{d}W_t, \quad t\in[0,T],\\ Y_T &= \delta_q\mathcal{G}(q_T,X_T^{\psi}).\end{cases} \tag{143}$$

Notice that this BSDE is not the one appearing in the first-order system ($\mathrm{Opt}_{\mathrm{N}}$), since at this stage of the proof, the relationship (23) between $(Y,Z)$ and $(Y^\star,Z^\star)$ is not yet known. The additional property (23) forms part of the necessary condition and is shown to hold under the assumption that $q\in\mathcal{Q}_{c_0}$ for some $c_0\in(0,c_1)$, with $q$ being a maximizer of $q'\mapsto\mathcal{J}(q')$ over $\mathcal{Q}_{c_1}$; see Lemma 28. For the time being, we establish the following well-posedness result:

**Lemma 27.** *There exists a unique solution* $(Y, Z) \in D(\mathbb{F}, \mathbb{Q}) \times (\cap_{\beta \in (0,1)} M^\beta(\mathbb{Q}, \mathbb{F}, \mathbb{R}^d))$ *to* (143)*. It is given by the formula (with the shorthand notation* $\delta_q \mathcal{G}(q_T)$ *in place of* $\delta_q \mathcal{G}(q_T, X_T^\psi)$*):*

$$Y_t = q_t^{-1} \mathbb{E}\left[ q_T \delta_q \mathcal{G}(q_T) + \int_t^T q_s \left( \ell_s - f^\star(s, Y_s^\star, Z_s^\star) \right) \mathrm{d}s \,\middle|\, \mathcal{F}_t \right], \quad t \in [0, T]. \tag{144}$$

*Proof.* The proof is mostly taken from [22]. We give it for the sake of completeness. We recall that $q_t = \Lambda_t \mathcal{E}_t$, for any $t \in [0, T]$, where $(\Lambda_t := \exp(\int_0^t Y_s^\star \mathrm{d}s))_{t \in [0,T]}$ and $(\mathcal{E}_t := \mathcal{E}_t(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s))_{t \in [0,T]}$.

*Step 1:* We claim that there exists a pair $(\tilde{Y}, \tilde{Z}) \in D(\mathbb{F}, \mathbb{Q}) \times (\cap_{\beta \in (0,1)} M^\beta(\mathbb{Q}, \mathbb{F}, \mathbb{R}^d))$ such that, for every $t \in [0, T]$,

$$\tilde{Y}_t = \Lambda_T \delta_q \mathcal{G}(q_T) + \int_t^T \Lambda_s \left( \ell_s - f^\star(s, Y_s^\star, Z_s^\star) \right) \mathrm{d}s - \int_t^T \tilde{Z}_s \cdot \mathrm{d}\tilde{W}_s, \tag{145}$$

where $(\tilde{W}_t := W_t - \int_0^t Z_s^\star \mathrm{d}s)_{t \in [0,T]}$ is a Brownian motion under the equivalent probability measure $\mathbb{Q} = \mathcal{E}_T \mathbb{P}$. Existence of the pair $(\tilde{Y}, \tilde{Z})$ is proven in two steps.

Throughout, the letter $C$ denotes a generic constant that only depends on the assumptions listed in Subsection 3.1 and that is, in particular, independent of $q$ and $\bar{\psi}$. The first observation is that

$$\mathbb{E}^\mathbb{Q}\left[ \Lambda_T |\delta_q \mathcal{G}(q_T)| + \int_0^T \Lambda_s |\ell_s - f^\star(s, Y_s^\star, Z_s^\star)| \mathrm{d}s \right] < +\infty, \tag{146}$$

which is a consequence of the following three bounds. First, by the growth Assumption A6, we have

$$\mathbb{E}^\mathbb{Q}\left[ \Lambda_T \left| \delta_q \mathcal{G}(q_T) \right| \right] = \mathbb{E}\left[ q_T \left| \delta_q \mathcal{G}(q_T) \right| \right] \le L \exp(\alpha T) \left( 1 + 2\mathbb{E}\left[ q_T |X_T^{\bar{\psi}}|^{2-r} \right] \right).$$

By Lemma 40, the last term satisfies the inequality

$$\mathbb{E}\left[ q_T \left| X_T^{\bar{\psi}} \right|^{2-r} \right] \le C \left( 1 + \mathcal{S}(q) + \mathcal{S}^\star(\bar{\psi}) \right).$$

Recalling that $q \in \mathcal{Q}_{c_1}$ and $\bar{\psi} \in \mathcal{A}_{c_2}$, we obtain

$$\mathbb{E}^\mathbb{Q}\left[ \Lambda_T \left| \delta_q \mathcal{G}(q_T) \right| \right] < +\infty.$$

By A4 and a direct application of the duality inequality (14), we also have

$$\mathbb{E}^\mathbb{Q}\left[ \int_0^T \Lambda_s |\ell_s| \mathrm{d}s \right] \le C + C\mathbb{E}\left[ \int_0^T q_s |\bar{\psi}_s|^2 \mathrm{d}s \right] \le C + C \left( \mathcal{S}(q) + \mathcal{S}^\star(\bar{\psi}) \right) < +\infty.$$

It remains to observe from 2 that $f^\star(t, y^\star, z^\star) \ge -f_t^0 := -f(t, 0, 0)$, which implies $|f^\star(t, y^\star, z^\star)| \le f^\star(t, y^\star, z^\star) + |f_t^0| + f_t^0$, and then

$$\begin{aligned} \mathbb{E}^\mathbb{Q}\left[ \int_0^T \Lambda_s |f^\star(s, Y_s^\star, Z_s^\star)| \mathrm{d}s \right] &\le \mathbb{E}\left[ \int_0^T q_s (f^\star(s, Y_s^\star, Z_s^\star) + f_s^0 + |f_s^0|) \mathrm{d}s \right] \\ &\le C \left( 1 + \mathcal{S}(q) \right) < +\infty. \end{aligned} \tag{147}$$

The last three displays imply (146).

As announced, we now follow [22, Section 6]. To do so, we consider $(\xi^k, \ell^k)_{k\in\mathbb{N}}$ such that, for each $k \in \mathbb{N}$, $\xi^k$ is a bounded $\mathcal{F}_T$-measurable random variable and $\ell^k = (\ell^k_t)_{t\in[0,T]}$ is a bounded $\mathbb{F}$-progressively measurable with the property that

$$\lim_{k\to+\infty} \mathbb{E}^{\mathbb{Q}} \left[ \Lambda_T |\xi^k - \delta_q \mathcal{G}(q_T)| + \int_0^T \Lambda_s |\ell^k_s - (\ell_s - f^\star(s, Y^\star_s, Z^\star_s))| \mathrm{d}s \right] = 0.$$

Then, for each $k \in \mathbb{N}$, we can define $(\tilde{Y}^k, \tilde{Z}^k)$ such that (we recall from [2, Theorem 2.4] that the martingale representation theorem holds under $\mathbb{Q}$, with respect to $\tilde{W}$)

$$\tilde{Y}^k_t = \Lambda_T \xi^k + \int_t^T \Lambda_s \ell^k_s \mathrm{d}s - \int_t^T \tilde{Z}^k_s \cdot \mathrm{d}\tilde{W}_s, \quad t \in [0,T].$$

There is no difficulty to see that

$$\tilde{Y}^k_t = \mathbb{E}^{\mathbb{Q}} \left[ \Lambda_T \xi^k + \int_t^T \Lambda_s \ell^k_s \mathrm{d}s \middle| \mathcal{F}_t \right], \quad t \in [0,T].$$

Since the term inside the conditional expectation appearing in the right-hand side is bounded, for each $k \in \mathbb{N}$, uniformly in $t \in [0,T]$, we easily deduce that the process $(\tilde{Y}^k, \tilde{Z}^k)$ satisfies the conclusion of the statement. In fact, item 2 is even satisfied in a stronger sense, as $\beta$ can be taken in $(0,+\infty)$.

The key step is to prove that the sequences $(\tilde{Y}^k)_{k\in\mathbb{N}}$ and $(\tilde{Z}^k)_{k\in\mathbb{N}}$ are Cauchy sequences in well-chosen spaces. As for $(\tilde{Y}^k)_{k\in\mathbb{N}}$, we notice that, for any $\mathbb{F}$-stopping time $\tau$ with values in $[0,T]$, for any $m, k \in \mathbb{N}$,

$$\mathbb{E}^{\mathbb{Q}} \left[ |\tilde{Y}^k_\tau - \tilde{Y}^m_\tau| \right] \le \mathbb{E}^{\mathbb{Q}} \left[ \Lambda_T |\xi^k - \xi^m| + \int_0^T \Lambda_s |\ell^k_s - \ell^m_s| \mathrm{d}s \right],$$

and the right-hand side tends to 0, as $m$ and $k$ tend to $+\infty$. This shows that

$$\lim_{N\to\infty} \sup_{m,k\ge N} \sup_\tau \mathbb{E}^{\mathbb{Q}} \left[ |\tilde{Y}^k_\tau - \tilde{Y}^m_\tau| \right] = 0,$$

with $\tau$ in the left-hand side being implicitly understood as a generic stopping time with values in $[0,T]$. Then, the analysis carried out in [22] (together with the references cited therein) permits us to show that there exists a process $\tilde{Y}$ satisfying item 1 in the statement such that

$$\lim_{k\to\infty} \sup_\tau \mathbb{E}^{\mathbb{Q}} \left[ |\tilde{Y}^k_\tau - \tilde{Y}_\tau| \right] = 0.$$

It then remains to handle the martingale integrand $(\tilde{Z}^k)_{k\in\mathbb{N}}$. Writing

$$\int_0^t \left( \tilde{Z}^k_s - \tilde{Z}^m_s \right) \cdot \mathrm{d}\tilde{W}_s = \left( \tilde{Y}^k_t - \tilde{Y}^m_t \right) - \left( \tilde{Y}^k_0 - \tilde{Y}^m_0 \right) - \int_0^t \Lambda_s \left( \ell^k_s - \ell^m_s \right) \mathrm{d}s,$$

for $t \in [0,T]$, we deduce from [22, Lemma 6.1] that, for any $\beta \in (0,1)$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}} \left[ \sup_{t\in[0,T]} \left| \int_0^t \left( \tilde{Z}^k_s - \tilde{Z}^m_s \right) \cdot \mathrm{d}\tilde{W}_s \right|^\beta \right] \\
&\le C_\beta \mathbb{E}^{\mathbb{Q}} \left[ \left| \int_0^T \left( \tilde{Z}^k_s - \tilde{Z}^m_s \right) \cdot \mathrm{d}\tilde{W}_s \right| \right]^\beta \\
&\le C_\beta \left( \mathbb{E}^{\mathbb{Q}} \left[ \Lambda_T |\xi^k - \xi^m| \right] + \mathbb{E}^{\mathbb{Q}} \left[ |Y^k_0 - Y^m_0| \right] + \mathbb{E} \left[ \int_0^T \Lambda_s |\ell^k_s - \ell^m_s| \mathrm{d}s \right] \right)^\beta,
\end{aligned}$$

for a constant $C_\beta > 0$ only depending on $\beta$. As a consequence, we obtain

$$\lim_{N\to\infty} \sup_{m,k\geq N} \mathbb{E}^{\mathbb{Q}} \left[ \sup_{t\in[0,T]} \left| \int_0^t \left( \tilde{Z}_s^k - \tilde{Z}_s^m \right) \cdot \mathrm{d}\tilde{W}_s \right|^\beta \right] = 0.$$

And then, by Bürkholder-Davis-Gundy inequality, it holds

$$\lim_{N\to\infty} \sup_{m,k\geq N} \mathbb{E}^{\mathbb{Q}} \left[ \left( \int_0^T |\tilde{Z}_s^k - \tilde{Z}_s^m|^2 \mathrm{d}s \right)^{\beta/2} \right] = 0,$$

and the existence of $\tilde{Z}$ as in the statement follows from a new application of Cauchy's convergence criterion in complete spaces.

*Step 2:* We now establish uniqueness of the pair $(Y, Z)$. Multiplying any solution by $(\Lambda_t)_{t\in[0,T]}$, uniqueness of the pair $(Y, Z)$ is in fact equivalent to uniqueness of the pair $(\tilde{Y}, \tilde{Z})$ in the expansion (145) (within the same space as in the statement).

By uniform integrability of the collection $(\tilde{Y}_\tau)_\tau$, when $\tau$ runs over the set of $\mathbb{F}$-stopping times with values in $[0, T]$ and by a standard localization argument, we deduce that, necessarily,

$$\tilde{Y}_t = \mathbb{E}^{\mathbb{Q}} \left[ \Lambda_T \delta_q \mathcal{G}(q_T) + \int_t^T \Lambda_s \left( \ell_s - f^\star(s, Y_s^\star, Z_s^\star) \right) \mathrm{d}s \middle| \mathcal{F}_t \right], \quad t \in [0, T]. \tag{148}$$

This establishes the uniqueness of $\tilde{Y}$. We then rewrite the equation for $(\tilde{Y}, \tilde{Z})$ in the form

$$\tilde{Y}_t - \int_0^t \Lambda_s (\ell_s - f^\star(s, Y_s^\star, Z_s^\star)) \mathrm{d}s = \int_0^t \tilde{Z}_s \cdot \mathrm{d}\tilde{W}_s, \quad t \in [0, T].$$

The right-hand side is a local martingale (by assumption). Since the left-hand side is given, we deduce that $\tilde{Z}$ is unique. Recalling the two formulas $\mathbb{Q} = \mathcal{E}_T(\int_0^\cdot Z_s^\star \mathrm{d}W_s)\mathbb{P}$ and $(q_t = \Lambda_t \mathcal{E}_t(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s))_{t\in[0,T]}$, we easily derive (144) from (148). □

At this stage, the notion of a solution to equation (143) is well defined and understood in the sense of Lemma 27. We are now in a position to establish the first-order condition for Nature.

**Lemma 28.** *For a given $c_0 \in (0, c_1)$, assume that there exists a maximizer $q \in \mathcal{Q}_{c_0}$ to the problem ($\mathrm{P}_{\mathrm{N},c_1}$) (the latter being set over $\mathcal{Q}_{c_1}$). Then, denoting by $(Y, Z)$ the solution of (143), the triple $(q, Y, Z)$ satisfies the first-order condition ($\mathrm{Opt}_{\mathrm{N}}$).*

*Proof. Step 1: localization procedure.* Generally speaking, our main objective is to prove that, for prescribed directions $(y^\star, z^\star) \in L^\infty(\mathbb{F}) \times L^\infty(\mathbb{F}, \mathbb{R}^d)$, $\mathbb{P}$-almost surely, for almost every $t \in [0, T]$,

$$f^\star(t, Y_t^\star + y_t^\star, Z_t^\star + z_t^\star) - f^\star(t, Y_t^\star, Z_t^\star) - (Y_t y_t^\star + Z_t \cdot z_t^\star) \geq 0. \tag{149}$$

From this, we will eventually derive (23) and then ($\mathrm{Opt}_{\mathrm{N}}$).

Although the proof of (149) follows seemingly standard arguments, it requires some non-trivial adjustments. We proceed by contradiction assuming that the left-hand side on (149) is negative on an event $E$ of positive measure under $\mathrm{Leb}_{[0,T]} \otimes \mathbb{P}$, namely

$$\begin{aligned} &\exists \varrho > 0, \quad \left( \mathrm{Leb}_{[0,T]} \otimes \mathbb{P} \right)(E) > 0, \\ &\text{with } E := \left\{ f^\star(t, Y_t^\star + y_t^\star, Z_t^\star + z_t^\star) - f^\star(t, Y_t^\star, Z_t^\star) - (Y_t y_t^\star + Z_t \cdot z_t^\star) \leq -\varrho \right\}. \end{aligned} \tag{150}$$

For a given $A > 0$, we also consider the stopping time

$$\tau_A \tag{151}$$
$$:= \inf\left\{t \in [0,T],\ \frac{1}{q_t} + \left|\int_0^t z_s^\star \cdot Z_s^\star \mathrm{d}s\right| + \left|\int_0^t z_s^\star \cdot Z_s \mathrm{d}s\right| + \left|\int_0^t z_s^\star \cdot \mathrm{d}W_s\right| \geq A\right\},$$

with the usual convention that the stopping time is equal to $+\infty$ if the set inside the infimum is empty. It is easy to prove that

$$\lim_{A\to+\infty} \mathbb{P}\left(\{\tau_A \leq T\}\right) = 0,$$

from which we deduce that we can choose $A$ large enough such that

$$\left(\mathrm{Leb}_{[0,T]} \otimes \mathbb{P}\right)\left(\{(t,\omega) \in E,\ t \leq \tau_A(\omega)\}\right) > 0. \tag{152}$$

We then define the new 'localized' directions

$$y_s^{\star,E}(\omega) := y_s^\star(\omega)\mathbb{1}_E(s,\omega), \quad z_s^{\star,A,E}(\omega) := z_s^\star(\omega)\mathbb{1}_E(s,\omega)\mathbb{1}_{[0,\tau_A(\omega)]}(s), \tag{153}$$

for $s \in [0,T]$ and $\omega \in \Omega$. For an intensity $\varepsilon \in (0,1]$, we consider the solution $(q_t^\varepsilon := \Lambda_t^\varepsilon \mathcal{E}_t^\varepsilon)_{t\in[0,T]}$ of the equation (4) driven by the pair

$$(Y^{\star,\varepsilon}, Z^{\star,\varepsilon}) := (Y^\star, Z^\star) + \varepsilon(y^{\star,E}, z^{\star,A,E})$$

and where

$$\Lambda_t^\varepsilon = \exp\left(\int_0^t Y_s^{\star,\varepsilon}\mathrm{d}s\right), \quad \mathcal{E}_t^\varepsilon = \mathcal{E}_t\left(\int_0^\cdot Z_s^{\star,\varepsilon} \cdot \mathrm{d}W_s\right), \quad t \in [0,T].$$

(For simplicity, we omit to precise the dependence on $A$ and $E$.)

We also introduce the process $q'$ which will be proved to be the variational process of $q$ in the direction $(y^{\star,E}, z^{\star,A,E})$. It is defined as

$$q_t' := q_t\left(\int_0^t \left(y_s^{\star,E} - Z_s^\star \cdot z_s^{\star,A,E}\right)\mathrm{d}s + \int_0^t z_s^{\star,A,E} \cdot \mathrm{d}W_s\right), \quad t \in [0,T]. \tag{154}$$

Using the definition of $z^{\star,A,E}$, we can check that, for all $t \in [0,T]$, $|q_t'| \leq Cq_t$, for a constant $C$ independent of $\varepsilon$ and $t$. Moreover, $q'$ solves the equation

$$q_t' = \int_0^t \left(q_s'Y_s^\star + q_s y_s^{\star,E}\right)\mathrm{d}s + \int_0^t \left(q_s'Z_s^\star + q_s z_s^{\star,A,E}\right)\cdot \mathrm{d}W_s, \quad t \in [0,T]. \tag{155}$$

We then let

$$\Delta q_s^\varepsilon := \varepsilon^{-1}\delta q_s^\varepsilon - q_s', \qquad \text{with} \quad \delta q_s^\varepsilon := q_s^\varepsilon - q_s,$$

$$\delta f_s^{\star,\varepsilon} := f_s^{\star,\varepsilon} - f_s^\star, \qquad \text{with} \quad \left\{\begin{array}{l} f_s^\star := f^\star\left(s, Y_s^\star, Z_s^\star\right), \\ f_s^{\star,\varepsilon} := f^\star\left(s, Y_s^{\star,\varepsilon}, Z_s^{\star,\varepsilon}\right), \end{array}\right.$$

for any $s \in [0,T]$. We observe that, at this stage, $f^{\star,\varepsilon}$ may take the value $+\infty$.

*Step 2: $q^\varepsilon \in \mathcal{Q}_{c_1}$ for $\varepsilon$ small enough.* We show that $q^\varepsilon$ is an admissible controlled process for ($\mathrm{P}_{\mathrm{N},c_1}$). The proof relies on the explicit formula for $q^\varepsilon$. For each time $t \in [0,T]$, we have

$$q_t^\varepsilon = q_t \exp\left(\varepsilon\left(\int_0^t y_s^{\star,E}\mathrm{d}s + \int_0^t z_s^{\star,A,E}\cdot \mathrm{d}W_s - \int_0^t \left(Z_s^\star \cdot z_s^{\star,A,E} + \tfrac{1}{2}\varepsilon|z_s^{\star,A,E}|^2\right)\mathrm{d}s\right)\right)$$
$$=: q_t \exp\left(\varepsilon\varphi_t^\varepsilon\right).$$

We can find a constant $C$, independent of $\varepsilon$, such that, with probability 1, $\sup_{t\in[0,T]}|\varphi_t^\varepsilon| \le C$. This follows from the fact that $y^\star$ and $z^\star$ are bounded and from the definitions of $\tau_A$, $y^{\star,E}$ and $z^{\star,A,E}$ (see (151) and (153)). And then, for any $\varepsilon \in (0,1]$ and any $t \in [0,T]$,

$$|q_t^\varepsilon - q_t| \le C\varepsilon\exp(C)q_t \le C\varepsilon\exp(C)\sup_{s\in[0,T]} q_s. \tag{156}$$

To establish the desired result, we expand the term $\mathcal{S}(q^\varepsilon)$ as

$$\mathcal{S}(q^\varepsilon) = \mathbb{E}\left[\int_0^T q_s f_s^{\star,\varepsilon}\mathrm{d}s\right] + \mathbb{E}\left[\int_0^T \delta q_s^\varepsilon f_s^{\star,\varepsilon}\mathrm{d}s\right]. \tag{157}$$

We study the two terms on the right-hand side separately. We start with the first one. By convexity of $f^\star$ in the variables $y^\star$ and $z^\star$ and by definition of the set $E$, see (150) and (152), we have (because $\varepsilon \in (0,1]$)

$$\begin{aligned}
&\frac{1}{\varepsilon}\delta f_s^{\star,\varepsilon}(\omega)\\
&\le \frac{1}{\varepsilon}\mathbb{1}_E(s,\omega)\mathbb{1}_{[0,\tau_A(\omega)]}(s)\left(f^\star\left(s, Y_s^\star + \varepsilon y_s^\star, Z_s^\star + \varepsilon z_s^\star\right) - f^\star\left(s, Y_s^\star, Z_s^\star\right)\right)\\
&\le \mathbb{1}_E(s,\omega)\mathbb{1}_{[0,\tau_A(\omega)]}(s)\left(y_s^\star Y_s^\star + z_s^\star \cdot Z_s^\star - \varrho\right)\\
&= y_s^{\star,E}Y_s^\star + z_s^{\star,A,E}\cdot Z_s^\star - \varrho\mathbb{1}_E(s,\omega)\mathbb{1}_{[0,\tau_A(\omega)]}(s),
\end{aligned} \tag{158}$$

for the same real $\varrho \ge 0$ as in (150). Multiplying both sides by $q$ and $\varepsilon$, adding $q_s f_s^\star$ on both sides, integrating from 0 to $T$ and using the fact that $\sup_{t\in[0,\tau_A]}|\int_0^t z_s^\star \cdot Z_s \mathrm{d}s| \le A$, we deduce that (allowing the value of $C$ to vary from line to line as long as it remains independent of $\varepsilon$)

$$\mathbb{E}\left[\int_0^T q_s f_s^{\star,\varepsilon}\mathrm{d}s\right] \le C\varepsilon + \mathcal{S}(q).$$

We now turn to the second term on the right-hand side of (157). From the inequality (156), we directly deduce that

$$\mathbb{E}\left[\int_0^T \delta q_s^\varepsilon f_s^{\star,\varepsilon}\mathrm{d}s\right] \le C\varepsilon\mathbb{E}\left[\int_0^T q_s|f_s^{\star,\varepsilon}|\mathrm{d}s\right] \le C\varepsilon + C\varepsilon\mathbb{E}\left[\int_0^T q_s f_s^{\star,\varepsilon}\mathrm{d}s\right] \le C\varepsilon,$$

where we used the bound $|f_t^{\star,\varepsilon}| \le f_t^{\star,\varepsilon} + C$, see (147), the value of the constant $C$ being allowed to change from one term to another.

Finally, plugging the last two estimates into (157) yields the desired result provided that $\varepsilon$ is small enough.

*Step 3: strong convergences of $\delta q^\varepsilon$ and $\Delta q^\varepsilon$.* As a direct consequence of the inequality (156), we deduce that, for $A$ fixed, $\mathbb{P}$ almost surely, $\sup_{t\in[0,T]}|q_t^\varepsilon - q_t| \to 0$ as $\varepsilon \to 0$.

Below, we also establish the almost sure uniform convergence of $\Delta q^\varepsilon$ to 0. To do so, one can refine the argument presented in Step 2 and provide a second-order (in $\varepsilon$) expansion of $q_t^\varepsilon$, writing, for all $t \in [0,T]$,

$$|q_t^\varepsilon - q_t(1+\varepsilon\varphi_t^\varepsilon)| \leq q_t \left|\exp(\varepsilon\varphi_t^\varepsilon) - (1+\varepsilon\varphi_t^\varepsilon)\right| \leq C^2\varepsilon^2 \exp(C) q_t. \tag{159}$$

Now, we use the fact that $(q_t\varphi_t^\varepsilon)_{t\in[0,T]}$ and $(q_t')_{t\in[0,T]}$ are close one from each other. Indeed,

$$\mathrm{d}\left[q_t\varphi_t^\varepsilon\right] = \Big(q_t\varphi_t^\varepsilon Y_t^\star + q_t y_t^{\star,E} - \tfrac{1}{2}\varepsilon|z_t^{\star,A,E}|^2\Big)\,\mathrm{d}t + \Big(q_t\varphi_t^\varepsilon Z_t^\star + q_t z_t^{\star,A,E}\Big)\cdot \mathrm{d}W_t, \quad t\in[0,T].$$

And then, thanks to (155),

$$\begin{aligned}
&\mathrm{d}\left[q_t' - q_t\varphi_t^\varepsilon\right] \\
&\quad = \Big(\left[q_t' - q_t\varphi_t^\varepsilon\right] Y_t^\star + \tfrac{1}{2}\varepsilon|z_t^{\star,A,E}|^2\Big)\,\mathrm{d}t + \Big(\left[q_t' - q_t\varphi_t^\varepsilon\right] Z_t^\star\Big)\cdot \mathrm{d}W_t, \quad t\in[0,T],
\end{aligned}$$

with 0 as initial condition. It is standard to deduce that

$$q_t' - q_t\varphi_t^\varepsilon = q_t\varepsilon\int_0^t q_s^{-1}|z_s^{\star,A,E}|^2\mathrm{d}s, \quad t\in[0,T].$$

Returning back to (159), we deduce from the above identity (together with the fact that $q_s^{-1} \leq A$ for $s \leq \tau_A$) that

$$\left|\delta q_t^\varepsilon - \varepsilon q_t'\right| \leq C\varepsilon^2 q_t, \quad t\in[0,T],$$

for a new value of $C$ (still independent of $\varepsilon$). Dividing by $\varepsilon$, we get

$$|\Delta q_t^\varepsilon| \leq C\varepsilon q_t, \quad t\in[0,T]. \tag{160}$$

*Step 4: derivative of the terminal and running costs.* In this step, we address the limit (as $\varepsilon$ tends to 0) of (as explained above, we omit the dependence on $\bar\psi$ in the notations)

$$\frac{1}{\varepsilon}\left(\mathcal{R}(q^\varepsilon) - \mathcal{R}(q)\right) = \frac{1}{\varepsilon}\left(\mathcal{G}(q_T^\varepsilon) - \mathcal{G}(q_T) + \mathbb{E}\left[\int_0^T \delta q_s^\varepsilon \ell_s \mathrm{d}s\right]\right). \tag{161}$$

We first compute the derivative of $\mathcal{G}$ along $q_T^\varepsilon$. We write

$$\begin{aligned}
&\frac{1}{\varepsilon}\left[\mathcal{G}(q_T^\varepsilon) - \mathcal{G}(q_T)\right] \\
&= \frac{1}{\varepsilon}\int_0^1 \mathbb{E}\left[\delta_q\mathcal{G}\left(q_T + \lambda\delta q_T^\varepsilon\right)\delta q_T^\varepsilon\right]\mathrm{d}\lambda \\
&= \int_0^1 \mathbb{E}\left[\delta_q\mathcal{G}\left(q_T + \lambda\delta q_T^\varepsilon\right) q_T'\right]\mathrm{d}\lambda + \int_0^1 \mathbb{E}\left[\delta_q\mathcal{G}\left(q_T + \lambda\delta q_T^\varepsilon\right)\Delta q_T^\varepsilon\right]\mathrm{d}\lambda.
\end{aligned} \tag{162}$$

Here, we recall from A6 that

$$|\delta_q\mathcal{G}\left(q_T + \lambda\delta q_T^\varepsilon\right)| \leq L\left(1 + \mathbb{E}\left[(q_T + \lambda\delta q_T^\varepsilon)|X_T|^{2-r}\right] + |X_T|^{2-r}\right).$$

Recalling that $|\delta q_t^\varepsilon| \le C\varepsilon q_t$ by (156), we deduce from Lemma 40 that the expectation on the right-hand side is bounded uniformly in $\varepsilon$ by

$$\mathbb{E}\left[(q_T + \lambda\delta q_T^\varepsilon)|X_T|^{2-r}\right] \le C\left(1 + \mathbb{E}\left[q_T|X_T|^{2-r}\right]\right) < +\infty.$$

Also, from the first step, we know that $q'_T \le Cq_T$. Then by Lemma 40 again, we deduce that $\mathbb{E}[(1+|X_T|^{2-r})q'_T] < +\infty$. By dominated convergence and by (156), we deduce that the first term on the right-hand side of (162) converges to

$$\lim_{\varepsilon\to 0}\int_0^1 \mathbb{E}\left[\delta_q\mathcal{G}\left(q_T + \lambda\delta q_T^\varepsilon\right)q'_T\right]\mathrm{d}\lambda = \mathbb{E}\left[\delta_q\mathcal{G}\left(q_T\right)q'_T\right].$$

As for the second term on the right-hand side of (162), we can proceed in the same way, but using in addition (160). We obtain

$$\lim_{\varepsilon\to 0}\int_0^1 \mathbb{E}\left[\delta_q\mathcal{G}\left(q_T + \lambda\delta q_T^\varepsilon\right)\Delta q_T^\varepsilon\right] = 0.$$

By combining the last two displays, we obtain

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\mathcal{G}(q_T^\varepsilon) - \mathcal{G}(q_T)\right) = \mathbb{E}\left[\delta_q\mathcal{G}\left(q_T\right)q'_T\right] = \mathbb{E}\left[Y_Tq'_T\right],$$

where we recall that $Y_T$ is the terminal condition given by the second line on (143).

We now turn to the second term in (161). Recalling that $\mathcal{S}^\star(\bar\psi) < +\infty$, we deduce from the bound (160) and the duality inequality (14) that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\left[\int_0^T \delta q_s^\varepsilon\ell_s\mathrm{d}s\right] = \mathbb{E}\left[\int_0^T q'_s\ell_s\mathrm{d}s\right].$$

Combining the last two limits, we obtain

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\mathcal{R}(q^\varepsilon) - \mathcal{R}(q)\right) = \mathbb{E}\left[Y_Tq'_T + \int_0^T q'_s\ell_s\mathrm{d}s\right]. \tag{163}$$

We then expand the first term on the right-hand side by means of Itô's formula, using the BSDE (143) satisfied by $(Y_t)_{t\in[0,T]}$ and the equation (155) satisfied by $q'$. We get for all $t\in[0,T]$,

$$\begin{aligned}\mathrm{d}\left[q'_tY_t\right] &= q'_t\left(-Y_t^\star Y_t - Z_t^\star\cdot Z_t + f^\star(t,Y_t^\star,Z_t^\star) - \ell_t\right)\mathrm{d}t\\ &\quad + Y_t\left(q'_tY_t^\star + q_ty_t^{\star,E}\right)\mathrm{d}t + Z_t\cdot\left(q'_tZ_t^\star + q_tz_t^{\star,A,E}\right)\mathrm{d}t + H_t\cdot\mathrm{d}W_t,\end{aligned}$$

with $(H_t := q'_tZ_t + Y_t(q'_tZ_t^\star + q_tz_t^{\star,A,E}))_{t\in[0,T]}$. Here, we need a new localization sequence to handle the local martingale. We define, for any $c>0$, $\varsigma_c := \inf\{t\in[0,T],\ |Y_t| + \int_0^t|Z_s|^2\mathrm{d}s \ge c\}$ (with $\inf\emptyset = +\infty$). Because $Y$ has continuous trajectories and $\mathbb{P}(\{\int_0^T|Z_t|^2\mathrm{d}t < +\infty\}) = 1$, it holds $\varsigma_c\to+\infty$ almost surely, as $c\to+\infty$. Then, recalling the identity $q'_0 = 0$ and the shorthand notation $f_t^\star$ for $f^\star(t,Y_t^\star,Z_t^\star)$, we can rearrange the above expansion and obtain

$$\begin{aligned}q'_{\varsigma_c\wedge T}Y_{\varsigma_c\wedge T} + \int_0^{\varsigma_c\wedge T} q'_s\ell_s\mathrm{d}s &= \int_0^{\varsigma_c\wedge T}\left(q'_sf_s^\star + q_sY_sy_s^{\star,E} + q_sZ_s\cdot z_s^{\star,A,E}\right)\mathrm{d}s\\ &\quad + \int_0^{\varsigma_c\wedge T} H_s\cdot\mathrm{d}W_s.\end{aligned}$$

Using the three bounds $|q'_t| \leq Cq_t$, for all $t \in [0,T]$, $\|z^{\star,A,E}\|_{L^\infty(\mathbb{F},\mathbb{R}^d)} < +\infty$ and $\mathbb{E}[q_T \int_0^T |Z_s^\star|^2 \mathrm{d}s] < +\infty$, together with the definition of the stopping time $\varsigma_c$, we can prove that $\mathbb{E}[\int_0^{\varsigma_c \wedge T} H_s \cdot \mathrm{d}W_s] = 0$, from which we deduce

$$\mathbb{E}\left[q'_{\varsigma_c \wedge T} Y_{\varsigma_c \wedge T} + \int_0^{\varsigma_c \wedge T} q'_s \ell_s \mathrm{d}s\right] = \mathbb{E}\left[\int_0^{\varsigma_c \wedge T} \left(q'_s f_s^\star + q_s Y_s y_s^{\star,E} + q_s Z_s \cdot z_s^{\star,A,E}\right) \mathrm{d}s\right]. \tag{164}$$

The point is to let $c$ tend to $+\infty$ on both sides. Thanks to the following three inequalities (the first line follows from the two bounds $|q'_t| \leq Cq_t$ and $|f_t^\star| \leq f_t^\star + C$, for $t \in [0,T]$, see (147)), the second one from the bound $\sup_{t\in[0,T]} \mathbb{E}[q_T |Y_t|] < +\infty$, see Lemma 27 and the definition of the space $D(\mathbb{F},\mathbb{Q})$ in Section 2, and the third one from (151),

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T |q'_s f_s^\star| \mathrm{d}s\right] \leq C\mathbb{E}\left[\int_0^T q_s f_s^\star \mathrm{d}s\right] + C = C\mathcal{S}(q) + C < +\infty, \\
&\mathbb{E}\left[\int_0^T q_s |Y_s y_s^\star| \mathrm{d}s\right] \leq C\mathbb{E}\left[q_T \int_0^T |Y_s| \mathrm{d}s\right] < +\infty, \\
&\mathbb{E}\left[\int_0^T q_s |Z_s \cdot z_s^{\star,A,E}| \mathrm{d}s\right] \leq \mathbb{E}\left[q_T \int_0^T |Z_s \cdot z_s^{\star,A,E}| \mathrm{d}s\right] \leq A\mathbb{E}\left[q_T\right] < +\infty,
\end{aligned}$$

we deduce, by dominated convergence theorem, that

$$\begin{aligned}
&\lim_{c\to+\infty} \mathbb{E}\left[\int_0^{\varsigma_c \wedge T} \left(q'_s f_s^\star + q_s Y_s y_s^{\star,E} + q_s Z_s \cdot z_s^{\star,A,E}\right) \mathrm{d}s\right] \\
&= \mathbb{E}\left[\int_0^T \left(q'_s f_s^\star + q_s Y_s y_s^{\star,E} + q_s Z_s \cdot z_s^{\star,A,E}\right) \mathrm{d}s\right].
\end{aligned}$$

Similarly,

$$\lim_{c\to+\infty} \mathbb{E}\left[\int_0^{\varsigma_c \wedge T} q'_s \ell_s \mathrm{d}s\right] = \mathbb{E}\left[\int_0^T q'_s \ell_s \mathrm{d}s\right].$$

It remains to pass to the limit in $\mathbb{E}[Y_{\varsigma_c \wedge T} q'_{\varsigma_c \wedge T}]$ as $c$ tends to $+\infty$. Almost surely, $Y_{\varsigma_c \wedge T} q'_{\varsigma_c \wedge T} \to Y_T q'_T$. Moreover, using the bound $|q'_t| \leq Cq_t$, we deduce that, for any event $F \in \mathcal{F}_T$, $\mathbb{E}[\mathbb{1}_F |Y_{\varsigma_c \wedge T} q'_{\varsigma_c \wedge T}|] \leq C\mathbb{E}^{\mathbb{Q}}[\mathbb{1}_F |Y_{\varsigma_c \wedge T}|]$. Recalling that the family $(Y_{\varsigma_c \wedge T})_{c>0}$ is uniformly integrable under $\mathbb{Q}$, see Lemma 27, we notice that the last term can be rendered as small as desired by choosing $\mathbb{Q}(F)$ small enough, and thus by choosing $\mathbb{P}(F)$ small enough. This shows that the collection $(|Y_{\varsigma_c \wedge T} q'_{\varsigma_c \wedge T}|)_{c>0}$ is uniformly integrable. Therefore, $\mathbb{E}[Y_{\varsigma_c \wedge T} q'_{\varsigma_c \wedge T}] \to \mathbb{E}[Y_T q'_T]$. Letting $c$ tend to $+\infty$ in (164), we get

$$\mathbb{E}\left[q'_T Y_T + \int_0^T q'_s \ell_s \mathrm{d}s\right] = \mathbb{E}\left[\int_0^T \left(q'_s f_s^\star + q_s Y_s y_s^{\star,E} + q_s Z_s \cdot z_s^{\star,A,E}\right) \mathrm{d}s\right].$$

And then returning back to (163), we get

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \left(\mathcal{R}(q^\varepsilon) - \mathcal{R}(q)\right) = \mathbb{E}\left[\int_0^T \left(q'_s f_s^\star + q_s \left(Y_s y_s^{\star,E} + Z_s \cdot z_s^{\star,A,E}\right)\right) \mathrm{d}s\right]. \tag{165}$$

*Step 5: sub-derivative of the entropic cost.* We now address the subgradient of $\varepsilon \mapsto \mathcal{S}(q^\varepsilon)$. By definition, we have (using the notation introduced in the first step)

$$\begin{aligned} \mathcal{S}(q^\varepsilon) - \mathcal{S}(q) &= \mathbb{E}\left[\int_0^T (q_s^\varepsilon f_s^{\star,\varepsilon} - q_s f_s^\star)\, \mathrm{d}s\right] \\ &= \mathbb{E}\left[\int_0^T q_s^\varepsilon \delta f_s^{\star,\varepsilon} \mathrm{d}s\right] + \mathbb{E}\left[\int_0^T \delta q_s^\varepsilon f_s^\star \mathrm{d}s\right], \end{aligned} \tag{166}$$

where, by construction,

$$\begin{aligned} \delta f_s^{\star,\varepsilon}(\omega) &= \mathbb{1}_E(s,\omega)\mathbb{1}_{[0,\tau_A(\omega)]}(s)\left(f^\star\left(s, Y_s^\star + \varepsilon y_s^{\star,E}, Z_s^\star + \varepsilon z_s^{\star,A,E}\right) - f^\star\left(s, Y_s^\star, Z_s^\star\right)\right) \\ &= \mathbb{1}_E(s,\omega)\mathbb{1}_{[0,\tau_A(\omega)]}(s)\left(f^\star\left(s, Y_s^\star + \varepsilon y_s^\star, Z_s^\star + \varepsilon z_s^\star\right) - f^\star\left(s, Y_s^\star, Z_s^\star\right)\right). \end{aligned}$$

By (158),

$$\frac{1}{\varepsilon}\delta f_s^{\star,\varepsilon}(\omega) \le y_s^{\star,E} Y_s^\star + z_s^{\star,A,E} \cdot Z_s^\star - \varrho \mathbb{1}_E(s,\omega)\mathbb{1}_{[0,\tau_A(\omega)]}(s),$$

for some the same real $\varrho \ge 0$ as in (150). Since the right-hand side is integrable under $\mathbb{Q}$, we deduce that (using the bound $q_s^\varepsilon \le C q_s$)

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}\mathbb{E}\left[\int_0^T q_s^\varepsilon \delta f_s^{\star,\varepsilon} \mathrm{d}s\right] \\ &\le \limsup_{\varepsilon\to 0} \mathbb{E}\left[\int_0^T q_s^\varepsilon \left(y_s^{\star,E} Y_s^\star + z_s^{\star,A,E} \cdot Z_s^\star\right) \mathrm{d}s\right] - \varrho \mathbb{E}\left[\int_0^T q_s^\varepsilon \mathbb{1}_E(s,\cdot)\mathbb{1}_{[0,\tau_A]}(s) \mathrm{d}s\right]. \\ &\le \mathbb{E}\left[\int_0^T q_s \left(y_s^{\star,E} Y_s^\star + z_s^{\star,A,E} \cdot Z_s^\star\right) \mathrm{d}s\right] - \varrho \mathbb{E}\left[\int_0^T q_s \mathbb{1}_E(s,\cdot)\mathbb{1}_{[0,\tau_A]}(s) \mathrm{d}s\right]. \end{aligned}$$

As for the second term on the right-hand side of (166), we know from (160) that $|\delta q_t^\varepsilon/\varepsilon - q_t'| \le C\varepsilon q_t$, which gives directly (using the bound $|f_s^\star| \le C + C f_s^\star$, see again (147))

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\mathbb{E}\left[\int_0^T \delta q_s^\varepsilon f_s^\star \mathrm{d}s\right] = \mathbb{E}\left[\int_0^T q_s' f_s^\star \mathrm{d}s\right].$$

Back to (166), we deduce that

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\mathcal{S}(q^\varepsilon) - \mathcal{S}(q)\right) \\ &\le \mathbb{E}\left[\int_0^T \left(q_s' f_s^\star + q_s(y_s^{\star,E} Y_s^\star + z_s^{\star,A,E} \cdot Z_s^\star) - \varrho q_s \mathbb{1}_E(s,\cdot)\mathbb{1}_{[0,\tau_A]}(s)\right) \mathrm{d}s\right]. \end{aligned}$$

Recalling from Lemma 18 that $q$ is strictly positive, we deduce from (150) and (152) that

$$\mathbb{E}\left[\int_0^T q_s \mathbb{1}_E(s,\cdot)\mathbb{1}_{[0,\tau_A]}(s) \mathrm{d}s\right] > 0,$$

and thus

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\mathcal{S}(q^\varepsilon) - \mathcal{S}(q)\right) < \mathbb{E}\left[\int_0^T \left(q_s' f_s^\star + q_s(y_s^{\star,E} Y_s^\star + z_s^{\star,A,E} \cdot Z_s^\star)\right) \mathrm{d}s\right]. \tag{167}$$

*Step 6: conclusion.* We now come back to the definition of $\mathcal{J}$, recalling that, for any $\varepsilon \in (0,1]$,

$$\frac{1}{\varepsilon}\left(\mathcal{J}(q^\varepsilon) - \mathcal{J}(q)\right) = \frac{1}{\varepsilon}\left(\mathcal{R}(q^\varepsilon) - \mathcal{R}(q)\right) - \frac{1}{\varepsilon}\left(\mathcal{S}(q^\varepsilon) - \mathcal{S}(q)\right). \tag{168}$$

Combining (165), (167) and (168), we obtain

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\mathcal{J}(q^\varepsilon) - \mathcal{J}(q)\right) > 0.$$

However, by optimality of $q$ and because $q^\varepsilon$ is admissible for ($\mathrm{P}_{\mathrm{N},c_1}$) for $\varepsilon$ small enough, we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\mathcal{J}(q^\varepsilon) - \mathcal{J}(q)\right) \leq 0,$$

which gives a contradiction with the penultimate line. This contradicts the assumption made in (150), as a result of which we deduce that, almost surely, for almost every $t \in [0,T]$,

$$f^\star(t, Y_t^\star + y_t^\star, Z_t^\star + z_t^\star) - f^\star(t, Y_t^\star, Z_t^\star) - (Y_t y_t^\star + Z_t \cdot z_t^\star) \geq 0.$$

By construction, the above holds true when the perturbation $(y^\star, z^\star)$ is bounded, but this assumption can be easily dropped by means of a truncation argument. In particular, we can choose $(y^\star, z^\star) = (\partial_y f(t, Y_t, Z_t) - Y_t^\star, \partial_z f(t, Y_t, Z_t) - Z_t^\star)_{t \in [0,T]}$. With this choice, we obtain (with full measure under $\mathrm{Leb}_{[0,T]} \otimes \mathbb{P}$)

$$\begin{aligned} &f^\star\left(t, \partial_y f(t, Y_t, Z_t), \partial_z f(t, Y_t, Z_t)\right) - f^\star(t, Y_t^\star, Z_t^\star) \\ &\geq Y_t \partial_y f(t, Y_t, Z_t) + Z_t \cdot \partial_z f(t, Y_t, Z_t) - Y_t Y_t^\star - Z_t \cdot Z_t^\star. \end{aligned}$$

Recalling that

$$\begin{aligned} &f(t, Y_t, Z_t) + f^\star\left(t, \partial_y f(t, Y_t, Z_t), \partial_z f(t, Y_t, Z_t)\right) \\ &= Y_t \partial_y f(t, Y_t, Z_t) + Z_t \cdot \partial_z f(t, Y_t, Z_t), \end{aligned}$$

we deduce (again, with full measure under $\mathrm{Leb}_{[0,T]} \otimes \mathbb{P}$)

$$Y_t Y_t^\star + Z_t \cdot Z_t^\star = f^\star(t, Y_t^\star, Z_t^\star) + f(t, Y_t, Z_t),$$

which is known to imply $(Y_t^\star, Z_t^\star) = (\partial_y f(t, Y_t, Z_t), \partial_z f(t, Y_t, Z_t))$. □

We complement the necessary condition established in Lemma 28 with a lower bound on the process $Y$. This bound plays a key role in the sufficient condition proved later in Lemma 30. In fact, this bound is similar to the one obtained in [50, Theorem 2.1]. Unfortunately, the bound established in [50] only holds for one specific solution of the quadratic equation ($\mathrm{Opt}_\mathrm{N}$) (obtained by taking the limit on truncated equations); in the absence of uniqueness for the quadratic equation, it is not possible to apply [50] to our case.

**Lemma 29.** *For a given $c_0 \in (0, c_1)$, let $q \in \mathcal{Q}_{c_0}$ be a maximizer to the problem ($\mathrm{P}_{\mathrm{N},c_1}$) (with $\bar{\psi} \in \mathcal{A}_{c_2}$ being fixed) and $(Y, Z)$ be the solution of (143). Then, $\mathbb{P}$-a.s., for any $t \in [0,T]$,*

$$Y_t \geq \tilde{Y}_t,$$

*where* $(\tilde{Y}, \tilde{Z})$ *solves the BSDE*

$$\begin{cases} -\mathrm{d}\tilde{Y}_t = \Big(\ell_t + f(t,0,0) + \Big(\partial_y f(t,0,0)\tilde{Y}_t + \partial_z f(t,0,0)\cdot \tilde{Z}_t\Big)\Big)\,\mathrm{d}t \\ \qquad\quad - \tilde{Z}_t \cdot \mathrm{d}W_t, \quad t\in[0,T], \\ \;\tilde{Y}_T = \delta_q \mathcal{G}(q_T). \end{cases} \tag{169}$$

*In addition, for any other* $\tilde{q} \in \mathcal{Q}_{c_1}$*, the random variables* $(\tilde{q}_\tau Y_\tau^- := \tilde{q}_\tau \min(-Y_\tau, 0))_\tau$*, with* $\tau$ *running over the set of* $[0,T]$*-valued* $\mathbb{F}$*-stopping times, are uniformly integrable under* $\mathbb{P}$*.*

*Proof. Step 1.* Recall that $q$ denotes a maximizer to the problem ($\mathrm{P}_{\mathrm{N},c_1}$). Then, for a given $t\in(0,T]$ and an arbitrary event $E\in\mathcal{F}_t$, we define $q^{t,E}$ by letting

$$q_s^{t,E} := \begin{cases} q_s & \text{if } s\in[0,t], \\ q_s \mathbb{1}_{E^\complement} + q_t Q_{t,s}\mathbb{1}_E & \text{if } s\in(t,T], \end{cases}$$

where

$$Q_{t,s} := \exp\left(\int_t^s \partial_y f(r,0,0)\mathrm{d}r\right)\mathcal{E}_s\left(\int_t^\cdot \partial_z f(r,0,0)\cdot \mathrm{d}W_r\right), \quad s\in[t,T].$$

Equivalently, this means that

$$\mathrm{d}q_s^{t,E} = q_s^{t,E} Y_s^{t,E}\mathrm{d}s + q_s^{t,E} Z_s^{t,E}\cdot \mathrm{d}W_s, \quad s\in[0,T],$$

where

$$(Y_s^{t,E}, Z_s^{t,E}) := \begin{cases} (Y_s^\star, Z_s^\star) & \text{if } s\in[0,t], \\ (Y_s^\star, Z_s^\star)\mathbb{1}_{E^\complement} + (\partial_y f(s,0,0), \partial_z f(s,0,0))\,\mathbb{1}_E & \text{if } s\in(t,T]. \end{cases}$$

Using the fact that $(\partial_y f(r,0,0))_{r\in[0,T]}$ and $(\partial_z f(r,0,0))_{r\in[0,T]}$ are bounded, we easily deduce that, for any $p\geq 1$, there exists a (deterministic) constant $C_p$, independent of $t$, such that, $\mathbb{P}$-a.s,

$$\mathbb{E}\left[\sup_{s\in[t,T]} Q_{t,s}^p \,\middle|\, \mathcal{F}_t\right] \leq C_p,$$

from which we deduce that

$$\sup_{s\in[t,T]} \mathbb{E}\left[h\left(q_s^{t,E}\right)\right] < +\infty. \tag{170}$$

In fact, we claim that there exists $\delta_0>0$, possibly depending on $t$, such that, for $\mathbb{P}(E)\leq\delta_0$, the process $q^{t,E}$ belongs to $\mathcal{Q}_{c_1}$. Indeed, we have (using Fenchel-Legendre duality to get the last line)

$$\begin{aligned} \mathcal{S}(q^{t,E}) &= \mathbb{E}\left[\int_0^T q_s^{t,E} f^\star(s, Y_s^{t,E}, Z_s^{t,E})\mathrm{d}s\right] \\ &= \mathbb{E}\left[\int_0^t q_s f^\star(s, Y_s^\star, Z_s^\star)\mathrm{d}s\right] + \mathbb{E}\left[\int_t^T q_s \mathbb{1}_{E^\complement} f^\star(s, Y_s^\star, Z_s^\star)\mathrm{d}s\right] \\ &\quad + \mathbb{E}\left[\int_t^T q_t Q_{t,s}\mathbb{1}_E f^\star(s, \partial_y f(s,0,0), \partial_z f(s,0,0))\mathrm{d}s\right] \\ &\leq \mathcal{S}(q) - \mathbb{E}\left[\int_t^T q_t Q_{t,s}\mathbb{1}_E f_s^0 \mathrm{d}s\right]. \end{aligned}$$

And then, using again the fact that $(f^0_s = f(s,0,0))_{s\in[0,T]}$ is bounded and recalling that $q \in \mathcal{Q}_{c_1'}$ with $c_1' < c_1$, we deduce that there exists a constant $C > 0$, such that

$$\mathcal{S}(q^{t,E}) \leq \mathcal{S}(q) + C\mathbb{E}\left[q_t \mathbb{1}_E\right] \leq c_1' + C\mathbb{E}\left[q_t \mathbb{1}_E\right].$$

Then for $\mathbb{P}(E)$ small enough, $c_1' + C\mathbb{E}\left[q_t \mathbb{1}_E\right]$ is less than or equal to $c_1$, which implies that $q^{t,E}$ belongs to $\mathcal{Q}_{c_1}$.

*Step 2.* Throughout, we assume that $t$ is fixed (in $(0,T]$) and $\mathbb{P}(E) \leq \delta_0$, with $\delta_0$ as in the first step. Since $q^{t,E} \in \mathcal{Q}_{c_1}$, and by optimality of $q$ on $\mathcal{Q}_{c_0}$, we deduce that

$$\mathcal{G}(q_T) + \mathbb{E}\left[\int_0^T q_s \ell_s \mathrm{d}s\right] - \mathcal{S}(q) \geq \mathcal{G}(q^{t,E}_T) + \mathbb{E}\left[\int_0^T q^{t,E}_s \ell_s \mathrm{d}s\right] - \mathcal{S}(q^{t,E}). \tag{171}$$

It is clear that

$$\mathbb{E}\left[\int_0^T (q^{t,E}_s - q_s)\ell_s \mathrm{d}s\right] = \mathbb{E}\left[\mathbb{1}_E \int_t^T \left(q_t Q_{t,s} - q_s\right)\ell_s \mathrm{d}s\right], \tag{172}$$

and

$$\begin{aligned} &\mathcal{S}(q^{t,E}) - \mathcal{S}(q) \\ &= \mathbb{E}\left[\mathbb{1}_E \int_t^T \left[q^{t,E}_s f^*\left(s, \partial_y f(s,0,0), \partial_z f(s,0,0)\right) - q_s f^*\left(s, Y^\star_s, Z^\star_s\right)\right] \mathrm{d}s\right] \\ &= -\mathbb{E}\left[\mathbb{1}_E \int_t^T \left[q^{t,E}_s f(s,0,0) + q_s f^*\left(s, Y^\star_s, Z^\star_s\right)\right] \mathrm{d}s\right]. \end{aligned} \tag{173}$$

We now turn to the difference between the two boundary conditions in (171). From the regularity property A7 (for $\mathcal{G}$), we have

$$\mathcal{G}(q^{t,E}_T) = \mathcal{G}(q_T) + \mathbb{E}\left[\left(q^{t,E}_T - q_T\right)\delta_q\mathcal{G}(q_T)\right] + o\left(\mathbb{E}\left[(1 + |X^\psi_T|^{2-r})|q^{t,E}_T - q_T|\right]\right),$$

where $o(r)/r \to 0$ as $r$ tends to 0 (the rate being independent of $E$). Recalling that $q^{t,E}$ and $q$ respectively belong to $\mathcal{Q}_{c_1}$ and $\mathcal{Q}_{c_0}$, we know from the growth Assumption A6 on $\mathcal{G}$ and from Lemma 40 that the first expectation on the right-hand side is well defined. Applying once again Lemma 40, we deduce that second expectation is also well defined. By definition of $q^{t,E}$, we have

$$\begin{aligned} \mathcal{G}(q^{t,E}_T) = \mathcal{G}(q_T) &+ \mathbb{E}\left[\mathbb{1}_E \left(q_t Q_{t,T} - q_T\right)\delta_q\mathcal{G}(q_T)\right] \\ &+ o\left(\mathbb{E}\left[(1 + |X^\psi_T|^{2-r})|q^{t,E}_T - q_T|\right]\right). \end{aligned} \tag{174}$$

Putting together (171), (172), (173) and (174), we obtain

$$\begin{aligned} &\mathbb{E}\left[\mathbb{1}_E \left(q_t Q_{t,T} - q_T\right)\delta_q\mathcal{G}(q_T)\right] + \mathbb{E}\left[\mathbb{1}_E \int_t^T \left(q_t Q_{t,s} - q_s\right)\ell_s \mathrm{d}s\right] \\ &+ \mathbb{E}\left[\mathbb{1}_E \int_t^T \left(q_t Q_{t,s} f(s,0,0) + q_s f^*\left(s, Y^\star_s, Z^\star_s\right)\right)\mathrm{d}s\right] \\ &\leq o\left(\mathbb{E}\left[(1 + |X^\psi_T|^{2-r})|q^{t,E}_T - q_T|\right]\right). \end{aligned} \tag{175}$$

Now, we recall from (144) that

$$q_t Y_t = \mathbb{E}\left[q_T \delta_q\mathcal{G}(q_T) + \int_t^T q_s\left(\ell_s - f^\star(s, Y^\star_s, Z^\star_s)\right)\mathrm{d}s \,\middle|\, \mathcal{F}_t\right].$$

And then, we can rewrite (175) as

$$\mathbb{E}\left[\mathbb{1}_E q_t\left(Q_{t,T}\delta_q\mathcal{G}(q_T)-Y_t\right)\right]+\mathbb{E}\left[\mathbb{1}_E\int_t^T q_tQ_{t,s}\ell_s\mathrm{d}s\right]$$
$$+\mathbb{E}\left[\mathbb{1}_E\int_t^T Q_{t,s}f(s,0,0)\mathrm{d}s\right]\leq o\left(\mathbb{E}\left[(1+|X_T^\psi|^{2-r})|q_T^{t,E}-q_T|\right]\right).$$

Also, it is easy to see from (169) that $\tilde{Y}_t$ can be represented as

$$\tilde{Y}_t=\mathbb{E}\left[Q_{t,T}\delta_q\mathcal{G}(q_T)+\int_t^T Q_{t,s}\left(\ell_s+f(s,0,0)\right)\mathrm{d}s\,\middle|\,\mathcal{F}_t\right],\tag{176}$$

which gives

$$\mathbb{E}\left[\mathbb{1}_E q_t\left(\tilde{Y}_t-Y_t\right)\right]\leq o\left(\mathbb{E}\left[(1+|X_T^\psi|^{2-r})|q_T^{t,E}-q_T|\right]\right).$$

We notice that the expectation in the right-hand side can be rewritten as

$$\mathbb{E}\left[(1+|X_T^\psi|^{2-r})|q_T^{t,E}-q_T|\right]=\mathbb{E}\left[\mathbb{1}_E(1+|X_T^\psi|^{2-r})|q_T-q_tQ_{t,T}|\right].$$

Writing $o(r)=r\eta(r)$, with $\eta\geq 0$ and $\lim_{r\to 0}\eta(r)=0$, and letting $R_{t,T}:=(1+|X_T^\psi|^{2-r})|q_T-q_tQ_{t,T}|$, we get

$$\mathbb{E}\left[\mathbb{1}_E\left\{q_t\left(\tilde{Y}_t-Y_t\right)-R_{t,T}\eta\left(\mathbb{E}\left[\mathbb{1}_E R_{t,T}\right]\right)\right\}\right]\leq 0.\tag{177}$$

The above is true for a given $t\in(0,T]$, for any event $E\in\mathcal{F}_t$ satisfying $\mathbb{P}(E)\leq\delta_0$. The function $\eta$ is independent of $E$. Moreover, we notice from (170) (with $E=\Omega$ therein) and Lemma 40 that $\mathbb{E}[R_{t,T}]<+\infty$.

*Step 3.* We now argue by contradiction to prove that $Y_t\geq\tilde{Y}_t$. Assume indeed that, for some $\varepsilon>0$,

$$\pi:=\mathbb{P}\left(\left\{q_t(\tilde{Y}_t-Y_t)\geq\varepsilon\right\}\right)>0.$$

Then, we can find $A>0$ such that the event

$$E_0:=\left\{q_t(\tilde{Y}_t-Y_t)\geq\varepsilon\right\}\cap\{R_{t,T}\leq A\}$$

satisfies $\mathbb{P}(E_0)\geq\pi/2$. By a standard uniform integrability argument (using the fact that $\mathbb{E}[R_{t,T}]<+\infty$), notice also that there exists $\delta>0$ such that

$$\mathbb{P}(E)\leq\delta\Rightarrow\eta\left(\mathbb{E}\left[\mathbb{1}_E R_{t,T}\right]\right)\leq\frac{\varepsilon}{2A}.$$

Decompose now $E_0$ as

$$E_0=\cup_{k\in\mathbb{N}}\left(E_0\cap\{W_t\in I_k\}\right),$$

where $(I_k)_{k\in\mathbb{N}}$ is partition of $\mathbb{R}^d$ into Borel subsets such that $\mathbb{P}(\{W_t\in I_k\})<\delta\wedge\delta_0$ (with $\delta_0$ as in the first step), for each $k\in\mathbb{N}$.

Applying (177) with $E=E_{0,k}:=E_0\cap\{W_t\in I_k\}$ for a given $k\in\mathbb{N}$, we obtain

$$0\geq\mathbb{E}\left[\mathbb{1}_{E_{0,k}}\left(q_t\left(\tilde{Y}_t-Y_t\right)-R_{t,T}\eta\left(\mathbb{E}\left[\mathbb{1}_{E_{0,k}}|q_T-q_tQ_{t,T}|\right]\right)\right)\right]$$
$$\geq\mathbb{E}\left[\mathbb{1}_{E_{0,k}}\left(\varepsilon-A\frac{\varepsilon}{2A}\right)\right]=\frac{\varepsilon}{2}\mathbb{P}(E_{0,k}).$$

This proves that $\mathbb{P}(E_{0,k}) = 0$, for each $k \in \mathbb{N}$, and then $\mathbb{P}(E_0) = 0$, which contradicts the fact that $\pi > 0$. We deduce that

$$\forall \varepsilon > 0, \quad \mathbb{P}\left(\left\{q_t(\tilde{Y}_t - Y_t) \geq \varepsilon\right\}\right) = 0,$$

i.e.,

$$\mathbb{P}\left(\left\{q_t(\tilde{Y}_t - Y_t) \leq 0\right\}\right) = 1.$$

Since $\mathbb{P}(\{q_t > 0\}) = 1$, we deduce that $\mathbb{P}(\{Y_t \geq \tilde{Y}_t\})$. This holds true for any $t \in (0, T]$. By continuity of the two processes $Y$ and $\tilde{Y}$, we deduce that $\mathbb{P}$-a.s., for any $t \in [0, T]$, $Y_t \geq \tilde{Y}_t$, which proves the first claim in the statement.

*Step 4.* It remains to establish that, for any other $\tilde{q} \in \mathcal{Q}_{c_1}$ (with decomposition $(\tilde{Y}^\star, \tilde{Z}^\star)$ in (4)), the family $(\tilde{q}_\tau Y_\tau^-)_\tau$, with $\tau$ running over the set of $[0, T]$-valued $\mathbb{F}$-stopping, is uniformly integrable. From the comparison principle established in Step 3, we first notice that

$$0 \leq Y_t^- \leq \tilde{Y}_t^-,$$

for any $t \in [0, T]$. Then to show the desired property, we provide a lower bound for the process $\tilde{Y}$ appearing on the right-hand side (recall (169) for its definition, and (176) for its representation). We notice from the strong convexity of $\ell$, see A4, that $\ell + f(0,0)$ is lower bounded. There exists a constant $c_0 \in \mathbb{R}$ such that $\tilde{Y} \geq \tilde{Y}^0$, where the process $\tilde{Y}^0$ is defined as

$$\tilde{Y}_t^0 = \frac{1}{q_t^0} \mathbb{E}\left[q_T^0 \delta_q \mathcal{G}(q_T) \middle| \mathcal{F}_t\right] + c_0(T - t),$$

and where $q^0$ is the solution to (84). Then, the following inequality holds

$$0 \leq Y_t^- \leq \tilde{Y}_t^- \leq (\tilde{Y}_t^0)^-.$$

Now, let $E$ be an element of $\mathcal{F}_T$ and $\tau$ a stopping time with values in $[0, T]$. For $\tilde{q} \in \mathcal{Q}_{c_1}$ as above, we have

$$0 \leq \mathbb{E}\left[\mathbb{1}_E \tilde{q}_\tau Y_\tau^-\right] \leq \mathbb{E}\left[\mathbb{1}_E \tilde{q}_\tau (\tilde{Y}_\tau^0)^-\right], \tag{178}$$

and we are left to establish that the right-hand side is finite for any $E \in \mathcal{F}_T$, and can be made small with $\mathbb{P}(E)$, uniformly with respect to $\tau$. We have

$$\tilde{q}_\tau \tilde{Y}_\tau^0 = \frac{\tilde{q}_\tau}{q_\tau^0} \mathbb{E}\left[q_T^0 \delta_q \mathcal{G}(q_T) \middle| \mathcal{F}_\tau\right] + c_0 \tilde{q}_\tau (T - \tau).$$

We notice that the process $(\tilde{q}_t^{0,\tau})_{t \in [0,T]}$ defined by $\tilde{q}_t^{0,\tau} := \tilde{q}_t$, for $t \in [0, \tau]$, and $\tilde{q}_t^{0,\tau} := \tilde{q}_\tau q_t^0 / q_\tau^0$, for $t \in [\tau, T]$, lies in $\mathcal{Q}$. It is indeed the solution to

$$\mathrm{d}\tilde{q}_t^{0,\tau} = \tilde{q}_t^{0,\tau} \tilde{Y}_t^{\star,0,\tau} \mathrm{d}t + \tilde{q}_t^{0,\tau} \tilde{Z}_t^{\star,0,\tau} \cdot \mathrm{d}W_t, \quad t \in [0, T]; \quad \tilde{q}_0^{0,\tau} = 1,$$

where

$$(\tilde{Y}_t^{\star,0,\tau}, \tilde{Z}_t^{\star,0,\tau}) := \begin{cases} (\tilde{Y}_t^\star, \tilde{Z}_t^\star) & \text{if} \quad t \leq \tau, \\ (\partial_y f(t,0,0), \partial_z f(t,0,0)) & \text{if} \quad t \in (\tau, T]. \end{cases}$$

Following the first step, we deduce that

$$\sup_{\tau} \mathcal{S}\left(\tilde{q}^{0,\tau}\right) < +\infty. \tag{179}$$

Moreover,

$$\mathbb{E}\left[\mathbb{1}_E \tilde{q}_\tau \left(\tilde{Y}^0_\tau\right)^-\right] \leq \mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)\tilde{q}^{0,\tau}_T \left(\delta_q \mathcal{G}(q_T)\right)^-\right] + |c_0|T\mathbb{E}[\mathbb{1}_E \tilde{q}_\tau], \tag{180}$$

As for the second term on the right-hand side, we know from Lemma 39 that $\mathbb{E}[\tilde{q}^*_T] < +\infty$. Therefore, the second term on the right-hand side is finite and can be made small with $\mathbb{P}(E)$, uniformly with respect to $\tau$. We now address the first term on the right-hand side in (180). By the lower estimate A6 on $\delta_q\mathcal{G}$, we have

$$\mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)\tilde{q}^{0,\tau}_T \left(\delta_q \mathcal{G}(q_T)\right)^-\right] \leq L\mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)\tilde{q}^{0,\tau}_T \left(1 + |X_T| + \mathbb{E}\left[q_T|X_T|^{2-r}\right]\right)\right].$$

By Lemma 40, we know that $\mathbb{E}\left[q_T|X_T|^{2-r}\right] < +\infty$ . We deduce that there exists $C > 0$ such that

$$\mathbb{E}\left[\mathbb{1}_E \tilde{q}^{0,\tau}_T \left(\delta_q \mathcal{G}(q_T)\right)^-\right] \leq C\mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)\tilde{q}^{0,\tau}_T \left(1 + |X_T|\right)\right]. \tag{181}$$

Following (180), we already know that $\mathbb{E}[\mathbb{P}(E|\mathcal{F}_\tau)\tilde{q}^{0,\tau}_T] = \mathbb{E}[\mathbb{1}_E\tilde{q}_\tau]$ tends to 0 as $\mathbb{P}(E)$ tends to 0. The remainder of the proof is devoted to establishing the same result for $\mathbb{E}[\mathbb{P}(E \mid \mathcal{F}_\tau)\tilde{q}^{0,\tau}_T|X_T|]$. To this end, we distinguish between the two cases $r = 0$ and $r = 1$.

If $r = 0$, then by Cauchy-Schwarz inequality and Lemma 40, we have

$$\begin{aligned}
\mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)\tilde{q}^{0,\tau}_T|X_T|\right] &\leq \mathbb{E}\left[\tilde{q}^{0,\tau}_T \mathbb{P}\left(E|\mathcal{F}_\tau\right)\right]^{1/2} \mathbb{E}\left[\tilde{q}^{0,\tau}_T|X_T|^2\right]^{1/2} \\
&\leq C\mathbb{E}\left[\tilde{q}^{0,\tau}_T \mathbb{P}\left(E|\mathcal{F}_\tau\right)\right]^{1/2} \left(1 + \mathcal{S}\left(\tilde{q}^{0,\tau}\right) + \mathcal{S}^\star(\bar{\psi})\right)^{1/2} \\
&\leq C\mathbb{E}\left[\tilde{q}_\tau \mathbb{1}_E\right]^{1/2} \left(1 + \mathcal{S}\left(\tilde{q}^{0,\tau}\right) + \mathcal{S}^\star(\bar{\psi})\right)^{1/2},
\end{aligned} \tag{182}$$

where we used the inequality $\mathbb{E}[\tilde{q}^{0,\tau}_T|\mathcal{F}_\tau] \leq C\tilde{q}_\tau$ to get the last line. The first term on the last line can be handled as the last term on (180). Combining (178), (179), (180) and (182), we easily complete the proof.

If $r = 1$, we return back to (181). In comparison with (182), the only difficult comes from the stochastic integral in the definition of $X$, as its integrand grows up linearly in $\psi$, see A2. By Girsanov theorem, we write

$$\begin{aligned}
\mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)\tilde{q}^{0,\tau}_T \left|\int_0^T \sigma(t,\psi_t)\mathrm{d}W_t\right|\right] &\leq \mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)\tilde{q}^{0,\tau}_T \left|\int_0^T \sigma(t,\psi_t)\mathrm{d}\tilde{W}^{0,\tau}_t\right|\right] \\
&\quad + C\mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)q_\tau \left|\int_0^\tau \sigma(t,\psi_t)Z^\star_t\mathrm{d}t\right|\right] \\
&\quad + \mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)\tilde{q}^{0,\tau}_T \left|\int_\tau^T \sigma(t,\psi_t)\partial_z f(t,0,0)\mathrm{d}t\right|\right],
\end{aligned} \tag{183}$$

where $\tilde{W}^{0,\tau}$ is a Brownian motion under $\mathcal{E}_T(\int_0^\cdot \tilde{Z}^{\star,0,\tau}_s \mathrm{d}s)$, and where $C$ is a constant independent of $\tau$ (which arises because $\mathbb{E}[\tilde{q}^{0,\tau}_T]$ may not be equal to 1). We first provide an upper bound for the first term on the right-hand side. By Young's inequality, observe that, for any $\varepsilon \in (0,1]$,

$$\mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)\tilde{q}_T^{0,\tau}\left|\int_0^T \sigma(t,\psi_t)\mathrm{d}\tilde{W}_t^{0,\tau}\right|\right] \leq \frac{1}{\varepsilon}\mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)\tilde{q}_T^{0,\tau}\right] + \varepsilon\mathbb{E}\left[\tilde{q}_T^{0,\tau}\left|\int_0^T \sigma(t,\psi_t)\mathrm{d}\tilde{W}_t^{0,\tau}\right|^2\right].$$

Here,

$$\mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)\tilde{q}_T^{0,\tau}\right] \leq \mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)q_\tau\right] = \mathbb{E}\left[\mathbb{1}_E q_\tau\right] \leq \mathbb{E}\left[\mathbb{1}_E q_T^*\right],$$

and

$$\mathbb{E}\left[\tilde{q}_T^{0,\tau}\left|\int_0^T \sigma(t,\psi_t)\mathrm{d}\tilde{W}_t^{0,\tau}\right|^2\right] \leq C\mathbb{E}\left[\tilde{q}_T^{0,\tau}\int_0^T |\sigma(t,\psi_t)|^2\,\mathrm{d}t\right] \leq C\mathbb{E}\left[\tilde{q}_T^{0,\tau}\int_0^T \left(1+|\psi_t|^2\right)\mathrm{d}t\right].$$

By (14) and (179), the above right-hand side is finite, uniformly with respect to $\tau$. By combining the last three displays, we easily deduce that the first-term on the right-hand side of (183) tends to 0 as $\mathbb{P}(E)$ tends to 0, uniformly with respect to $\tau$. Using similar arguments together with the fact that $(\partial_z f(t,0,0))_{t\in[0,T]}$ is bounded, we can reach the same conclusion for the third term on the right-hand side of (183). It remains to handle the second term on the right-hand side of (183). To do so, it suffices to notice that

$$\mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)q_\tau\left|\int_0^\tau \sigma(t,\psi_t)Z_t^\star\mathrm{d}t\right|\right] \leq C\mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_\tau)q_T\left|\int_0^\tau \sigma(t,\psi_t)Z_t^\star\mathrm{d}t\right|\right] \leq C\mathbb{E}\left[\sup_{t\in[0,T]}\mathbb{P}(E|\mathcal{F}_t)q_T\int_0^T (1+|\psi_t|)\,|Z_t^\star|\mathrm{d}t\right]. \tag{184}$$

By Doob's inequality, $\sup_{t\in[0,T]}\mathbb{P}(E|\mathcal{F}_t)$ tends to 0 in probability as $\mathbb{P}(E)$ tends to 0. Moreover,

$$\mathbb{E}\left[q_T\int_0^T (1+|\psi_t|)\,|Z_t^\star|\mathrm{d}t\right] \leq C\mathbb{E}\left[q_T\int_0^T (1+|Z_t^\star|^2)\,\mathrm{d}t\right] + C\mathbb{E}\left[q_T\int_0^T (1+|\psi_t|^2)\,\mathrm{d}t\right].$$

Thanks to (14), the second term on the right-hand side is finite. By (102), the first term is also finite. This proves that the left-hand side on (184) tends to 0 as $\mathbb{P}(E)$ tends to 0, uniformly in $\tau$. □

#### 5.2.3 Sufficient condition

We now turn to the proof of the sufficient condition, i.e. the second assertion in the statement of Theorem 25. We recall that $\bar{\psi} \in \mathcal{A}_{c_2}$ is given. Also, we use the same abbreviated notations as in (142).

**Lemma 30.** *Assume that there exists a triple $(q, Y, Z) \in \mathscr{Q}$ satisfying the first order condition* ($\mathrm{Opt}_\mathrm{N}$). *Then, $q$ is the unique maximizer of the mapping $q' \in \mathcal{Q} \mapsto \mathcal{J}(q', \bar{\psi})$.*

*Proof.* Throughout the proof, we omit the dependence on $\bar{\psi}$ in the various notations. For instance, we just write $\mathcal{J}(q')$ for $\mathcal{J}(q', \bar{\psi})$.

Moreover, in addition to $(q, Y^\star, Z^\star)$, we let $(\tilde{q}, \tilde{Y}^\star, \tilde{Z}^\star)$ be another arbitrary tuple of state and control in $\mathscr{Q}$, and then denote $\delta q := \tilde{q} - q$. By definition of $\mathcal{J}$ and concavity of $\mathcal{G}$ with respect to its second variable $q$ (assumption (A8)), we have

$$\begin{aligned} \mathcal{J}(\tilde{q}) - \mathcal{J}(q) &= \mathcal{R}(\tilde{q}) - \mathcal{R}(q) - (\mathcal{S}(\tilde{q}) - \mathcal{S}(q)) \\ &\leq \mathbb{E}\left[\delta q_T \delta_q \mathcal{G}(q_T) + \int_0^T \delta q_t \ell_t \mathrm{d}t\right] - (\mathcal{S}(\tilde{q}) - \mathcal{S}(q)). \end{aligned} \tag{185}$$

Notice that the right-hand side is finite, which can be shown in the same way as in (146), using the duality inequality (14), Lemma 40 and the bound $\mathcal{S}(\tilde{q}) + \mathcal{S}(q) + \mathcal{S}^\star(\bar{\psi}) < +\infty$.

*Step 1: localization.* In this first step, we proceed as in the analysis of the right-hand side on (163) and expand $(\delta q_t Y_t)_{t\in[0,T]}$ by means of Itô's formula. Recalling (143), we have

$$\begin{aligned} \mathrm{d}\left[\delta q_t Y_t\right] = \delta q_t &\left(-Y_t^\star Y_t - Z_t^\star \cdot Z_t + f^\star(t, Y_t^\star, Z_t^\star) - \ell_t\right) \mathrm{d}t \\ &+ Y_t\left(\tilde{q}_t \tilde{Y}_t^\star - q_t Y_t^\star\right) \mathrm{d}t + Z_t \cdot \left(\tilde{q}_t \tilde{Z}_t^\star - q_t Z_t^\star\right) \mathrm{d}t \\ &+ \delta q_t Z_t \cdot \mathrm{d}W_t + Y_t\left(\tilde{q}_t \tilde{Z}_t^\star - q_t Z_t^\star\right) \cdot \mathrm{d}W_t, \quad t \in [0, T]. \end{aligned} \tag{186}$$

For a given $A > 0$, we then introduce the following stopping time

$$\tau_A := \inf\left\{t \in [0, T],\ |Y_t| + q_t + 1/q_t + \tilde{q}_t + \int_0^t |Z_s|^2 \mathrm{d}s \geq A\right\},$$

with the standard convention that $\tau_A = +\infty$ if the set on the right-hand side is empty. With this definition in hand, we notice that, for $t \in [0, \tau_A]$

$$\begin{aligned} |\delta q_t| &\leq A \leq A^2 q_t, \\ \tilde{q}_t &\leq A \leq A^2 q_t. \end{aligned}$$

This implies in particular that, for a constant $C_A$ depending on $A$,

$$\begin{aligned} &\mathbb{E}\left[\int_0^{\tau_A} \left|\delta q_t \left(-Y_t^\star Y_t - Z_t^\star \cdot Z_t + f^\star(t, Y_t^\star, Z_t^\star) - \ell_t\right)\right| \mathrm{d}t\right] \\ &\leq C_A \mathbb{E}\left[\int_0^{\tau_A} q_t \left(|Y_t^\star Y_t| + |Z_t^\star \cdot Z_t| + |f^\star(t, Y_t^\star, Z_t^\star)| + |\ell_t|\right) \mathrm{d}t\right] \\ &\leq C_A \left(1 + \mathbb{E}\left[\int_0^{\tau_A} q_t \left(|Y_t^\star| + |Z_t^\star|^2 + |f^\star(t, Y_t^\star, Z_t^\star)| + |\ell_t|\right) \mathrm{d}t\right]\right) \\ &\leq C_A \left(1 + \mathbb{E}\left[\int_0^{T} q_t \left(|Y_t^\star| + |f^\star(t, Y_t^\star, Z_t^\star)| + |\ell_t|\right) \mathrm{d}t\right]\right) < +\infty, \end{aligned}$$

with the third line following from the condition $|Y_t| + \int_0^t |Z_s|^2 \mathrm{d}s \leq A$, and the last line following from (16), and from the fact that $\mathcal{S}(q)$ and $\mathcal{S}^\star(\bar{\psi})$ are finite.

Similarly,

$$\mathbb{E}\left[\int_0^{\tau_A} \left|Y_t\left(\tilde{q}_t\tilde{Y}_t^\star - q_tY_t^\star\right)\right| \mathrm{d}t\right] \le C_A\mathbb{E}\left[\int_0^T q_t\left(|\tilde{Y}_t^\star| + |Y_t^\star|\right)\mathrm{d}t\right] < +\infty,$$
$$\mathbb{E}\left[\int_0^{\tau_A} \left|Z_t\cdot\left(\tilde{q}_t\tilde{Z}_t^\star - q_tZ_t^\star\right)\right| \mathrm{d}t\right] \le C_A\left(1+\mathbb{E}\left[\int_0^T q_t\left(|\tilde{Z}_t^\star|^2 + |Z_t^\star|^2\right)\mathrm{d}t\right]\right) < +\infty.$$

Back to (186), this proves that the terms on the first and second lines of the right-hand side, when integrated between 0 and $\tau_A$, have a finite expectation.

It remains to check in a similar way that the stochastic integrals on the third line of (186) have a zero expectation, when they are integrated between 0 and $\tau_A$. To do so, we notice that

$$\mathbb{E}\left[\int_0^{\tau_A} |\delta q_t|^2|Z_t|^2\mathrm{d}t\right] \le C_A\mathbb{E}\left[\int_0^{\tau_A} q_t|Z_t|^2\mathrm{d}t\right] < +\infty,$$
$$\mathbb{E}\left[\int_0^{\tau_A} |Y_t|^2|\tilde{q}_t\tilde{Z}_t^\star - q_tZ_t^\star|^2\mathrm{d}t\right] \le C_A\mathbb{E}\left[\int_0^T \left(\tilde{q}_t|\tilde{Z}_t^\star|^2 + q_t|Z_t^\star|^2\right)\mathrm{d}t\right] < +\infty.$$

Therefore, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\delta q_{T\wedge\tau_A}Y_{T\wedge\tau_A} + \int_0^{T\wedge\tau_A}\delta q_t\ell_t\mathrm{d}t\right]\\
&= \mathbb{E}\left[\int_0^{T\wedge\tau_A}\delta q_t\left(-Y_t^\star Y_t - Z_t^\star\cdot Z_t + f^\star(t,Y_t^\star,Z_t^\star)\right)\mathrm{d}t\right]\\
&\quad + \mathbb{E}\left[\int_0^{T\wedge\tau_A}\left(Y_t\left(\tilde{q}_t\tilde{Y}_t^\star - q_tY_t^\star\right) + Z_t\cdot\left(\tilde{q}_t\tilde{Z}_t^\star - q_tZ_t^\star\right)\right)\mathrm{d}t\right].
\end{aligned}$$

Introduce now the notations $(\tilde{f}_t^\star = \tilde{f}^\star(t,\tilde{Y}_t^\star,\tilde{Z}_t^\star))_{t\in[0,T]}$ and $(f_t^\star = f^\star(t,Y_t^\star,Z_t^\star))_{t\in[0,T]}$ and deduce that

$$\begin{aligned}
&\mathbb{E}\left[\delta q_{T\wedge\tau_A}Y_{T\wedge\tau_A} + \int_0^{T\wedge\tau_A}\delta q_t\ell_t\mathrm{d}t - \int_0^{T\wedge\tau_A}\left(\tilde{q}_t\tilde{f}_t^\star - q_tf_t^\star\right)\mathrm{d}t\right]\\
&= -\mathbb{E}\left[\int_0^{T\wedge\tau_A}\left(\tilde{q}_t(\tilde{f}_t^\star - f_t^\star - (\tilde{Y}_t^\star - Y_t^\star)Y_t - (\tilde{Z}_t^\star - Z_t^\star)\cdot Z_t)\right)\mathrm{d}t\right].
\end{aligned}$$

Then, by the first order condition ($\mathrm{Opt_N}$) and the (strict) joint convexity of $f^\star$, we deduce that the right-hand side is non-positive, i.e., for any $A > 0$,

$$\mathbb{E}\left[\delta q_{T\wedge\tau_A}Y_{T\wedge\tau_A} + \int_0^{T\wedge\tau_A}\delta q_t\ell_t\mathrm{d}t - \int_0^{T\wedge\tau_A}\left(\tilde{q}_t\tilde{f}_t^\star - q_tf_t^\star\right)\mathrm{d}t\right] \le 0. \tag{187}$$

The key step in the rest of the proof is to let $A$ tend to $+\infty$ on the left-hand side of (187). This requires some extra care due to the rather weak integrability properties of $\tilde{q}$ and $Y$. In order to proceed, we write the integrand in the form

$$\begin{aligned}
&\delta q_{T\wedge\tau_A}Y_{T\wedge\tau_A} + \int_0^{T\wedge\tau_A}\delta q_t\ell_t\mathrm{d}t - \int_0^{T\wedge\tau_A}\left(\tilde{q}_t\tilde{f}_t^\star - q_tf_t^\star\right)\mathrm{d}t\\
&= \tilde{q}_{T\wedge\tau_A}Y_{T\wedge\tau_A} - q_{T\wedge\tau_A}Y_{T\wedge\tau_A} + \int_0^{T\wedge\tau_A}\delta q_t\ell_t\mathrm{d}t - \int_0^{T\wedge\tau_A}\left(\tilde{q}_t\tilde{f}_t^\star - q_tf_t^\star\right)\mathrm{d}t\\
&=: T^1(A) + T^2(A) + T^3(A) + T^4(A).
\end{aligned} \tag{188}$$

*Step 2: limit $A \to +\infty$ in $T^2(A)$, $T^3(A)$ and $T^4(A)$.* We claim that each of the three families of random variables $(T^2(A))_{A>0}$, $(T^3(A))_{A>0}$ and $(T^4(A))_{A>0}$ is uniformly integrable. For $(T^2(A))_{A>0}$, we observe that there exists a constant $C \geq 0$ (whose value may change from line to line) such that, for any event $E \in \mathcal{F}_T$ and any constant $A' > 0$,

$$\begin{aligned}\mathbb{E}\left[|T^2(A)|\mathbb{1}_E\right] &\leq A'\mathbb{E}\left[q_{T\wedge\tau_A}\mathbb{1}_E\right] + \mathbb{E}\left[q_{T\wedge\tau_A}\mathbb{1}_{\{|Y_{T\wedge\tau_A}|\geq A'\}}|Y_{T\wedge\tau_A}|\right] \\ &\leq A'\mathbb{E}\left[q_{T\wedge\tau_A}\mathbb{1}_E\right] + C\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{|Y_{T\wedge\tau_A}|\geq A'\}}|Y_{T\wedge\tau_A}|\right],\end{aligned}$$

where $\mathbb{Q} := \mathcal{E}_T(\int_0^{\cdot} Z_t^\star \cdot \mathrm{d}W_t)\mathbb{P}$. Since $Y \in D(\mathbb{F}, \mathbb{Q})$, the family $(Y_{T\wedge\tau_A})_{A>0}$ is uniformly integrable under $\mathbb{Q}$, which shows that the last term tends to 0 as $A'$ tends to $+\infty$, uniformly in $A > 0$. Therefore, to establish the uniformly integrability of the family $(T^2(A))_{A>0}$, it suffices to recall that $\mathbb{E}[q_T^*] < +\infty$ (see Lemma 39), which implies in particular that

$$\lim_{\mathbb{P}(E)\to 0} \sup_{A>0} \mathbb{E}\left[q_{T\wedge\tau_A}\mathbb{1}_E\right] = 0.$$

We now establish the uniform integrability of $(T^3(A))_{A>0}$. By a standard domination argument, it suffices to notice that

$$\mathbb{E}\left[\int_0^T |\delta q_t \ell_t| \,\mathrm{d}t\right] \leq C\left(1 + \mathbb{E}\left[\int_0^T (q_t + \tilde{q}_t)\,|\bar{\psi}_t|^2 \mathrm{d}t\right]\right) < +\infty,$$

with the last inequality following from the fact that $q$ and $\tilde{q}$ belong to $\mathcal{Q}$ and $\bar{\psi}$ to $\mathcal{A}_{c_2}$ (together with the duality inequality (14)).

We handle $(T^4(A))_{A>0}$ in the same way. Indeed, by the same argument as in (147), we have

$$\mathbb{E}\left[\int_0^T \left(\tilde{q}_t|\tilde{f}_t^\star| + q_t|f_t^\star|\right)\mathrm{d}t\right] < +\infty.$$

Combining the uniform integrability properties of $(T^2(A))_{A>0}$, $(T^3(A))_{A>0}$ and $(T^4(A))_{A>0}$ together with the time continuity of the processes appearing in (188), we deduce that

$$\begin{aligned}&\lim_{A\to+\infty} \mathbb{E}\left[T^2(A) + T^3(A) + T^4(A)\right] \\ &= \mathbb{E}\left[-q_T Y_T + \int_0^T \delta q_t \ell_t \mathrm{d}t - \int_0^T \left(\tilde{q}_t \tilde{f}_t^\star - q_t f_t^\star\right)\mathrm{d}t\right].\end{aligned} \tag{189}$$

*Step 3: limit $A \to +\infty$ in $T^1(A)$.* We now explain how to handle $(T^1(A))_{A>0}$ in (188). We first decompose $T^1(A)$ into non positive and non negative parts

$$T^1(A) = \tilde{q}_{T\wedge\tau_A}\left(Y_{T\wedge\tau_A} + Y^-_{T\wedge\tau_A}\right) - \tilde{q}_{T\wedge\tau_A} Y^-_{T\wedge\tau_A}. \tag{190}$$

By Lemma 29, the family $(\tilde{q}_{T\wedge\tau_A} Y^-_{T\wedge\tau_A})_{A>0}$ is uniformly integrable. In particular,

$$\lim_{A\to+\infty} \mathbb{E}\left[\tilde{q}_{T\wedge\tau_A} Y^-_{T\wedge\tau_A}\right] = \mathbb{E}\left[\tilde{q}_T Y_T^-\right]. \tag{191}$$

We also notice that, $\tilde{q}_{T\wedge\tau_A}(Y_{T\wedge\tau_A} + Y^-_{T\wedge\tau_A})$ takes values in $[0, +\infty)$. Therefore, Fatou's lemma gives

$$\liminf_{A\to+\infty} \mathbb{E}\left[\tilde{q}_{T\wedge\tau_A}\left(Y_{T\wedge\tau_A} + Y^-_{T\wedge\tau_A}\right)\right] \geq \mathbb{E}\left[\tilde{q}_T\left(Y_T + Y_T^-\right)\right].$$

Inserting the latter into (190), we obtain

$$\liminf_{A\to+\infty} \mathbb{E}\left[T^1(A)\right] = \liminf_{A\to+\infty} \mathbb{E}\left[\tilde{q}_{T\wedge\tau_A} Y_{T\wedge\tau_A}\right] \geq \mathbb{E}\left[\tilde{q}_T\left(Y_T + Y_T^- - Y_T^-\right)\right] = \mathbb{E}\left[\tilde{q}_T Y_T\right].$$

And then, thanks to (189), this gives

$$\begin{aligned}
&\liminf_{A\to+\infty} \mathbb{E}\left[T^1(A) + T^2(A) + T^3(A) + T^4(A)\right] \\
&\geq \mathbb{E}\left[\tilde{q}_T Y_T - q_T Y_T + \int_0^T (\tilde{q}_t - q_t)\ell_t \mathrm{d}t - \int_0^T \left(\tilde{q}_t \tilde{f}_t^\star - q_t f_t^\star\right) \mathrm{d}t\right].
\end{aligned}$$

By (187), the left-hand side is less than 0, from which we deduce that

$$0 \geq \mathbb{E}\left[\tilde{q}_T Y_T - q_T Y_T + \int_0^T (\tilde{q}_t - q_t)\ell_t \mathrm{d}t - \int_0^T \left(\tilde{q}_t \tilde{f}_t^\star - q_t f_t^\star\right) \mathrm{d}t\right].$$

By (185), the right-hand side is greater than $\mathcal{J}(\tilde{q}) - \mathcal{J}(q)$. This shows $\mathcal{J}(\tilde{q}) - \mathcal{J}(q) \leq 0$, which proves the optimality of $q$.

Uniqueness of the minimizer follows from the strict convexity of $\mathcal{J}$ in the variable $q$, see Proposition 20. □

### 5.3 Central planner's control problem

In this section, we study the problem of the central planner,

$$\inf_{\psi\in\mathcal{A}_{c_2}} \mathcal{J}(\bar{q}, \psi), \qquad (\mathrm{P_C})$$

under the assumption that there exists $\bar{\psi}$ such that the pair $(\bar{q}, \bar{\psi}) \in \mathcal{Q}_{c_1} \times \mathcal{A}_{c_2}$ is a saddle point of the problem (P'). We recall (27) and (28) for the definitions of the two sets $\mathcal{Q}_{c_1}$ and $\mathcal{A}_{c_2}$. We further recall (25) for the definition of the set $\mathscr{A}$. The purpose of this section is to establish the following characterization of the problem ($\mathrm{P_C}$).

**Theorem 31.** *There exists a constant $c_2' > 0$, only depending on the data and $c_1$, such that, if $c_2 > c_2'$, the minimizer $\psi$ of ($\mathrm{P_C}$) (over $\mathcal{A}_{c_2}$) belongs in fact to $\mathcal{A}_{c_2'}$. Conversely, if $c_2 > c_2'$, any $\psi \in \mathcal{A}_{c_2'}$ is a minimizer to the problem ($\mathrm{P_C}$) over $\mathcal{A}_{c_2}$ if and only if there exists a tuple $(\psi, p, k, X) \in \mathscr{A}$ solving the FBSDE ($\mathrm{Opt_C}$) (with $q$ replaced by $\bar{q}$).*

*Sketch of Proof.* In Lemma 32 below, we establish the existence of a constant $c_2' > 0$, only depending on the data and $c_1$, independent of $c_2$, such that any minimizer $\psi$ to the problem ($\mathrm{P_C}$) over $\mathcal{A}_{c_2}$ is in fact in $\mathcal{A}_{c_2'}$. By assuming (without any loss of generality) that the constant $c_2$ is strictly larger than $c_2'$, we ensure that the minimizer $\psi$ is an interior solution in the sense that

$$\mathcal{S}^\star(\psi) < c_2.$$

The conclusion of the statement follows from the necessary condition proved in Lemma 37 and the sufficient condition established in Lemma 38. □

### 5.3.1 A priori estimate

This subsection is devoted to proving the following a priori estimate for the component $\bar{\psi}$ of the saddle point $(\bar{q}, \bar{\psi})$.

**Lemma 32.** *There exists a positive constant $c_2' > 0$, only depending on $c_1$ and on the data (and in particular independent of the parameter $c_2$), such that the component $\bar{\psi}$ of any saddle point $(\bar{q}, \bar{\psi}) \in \mathcal{Q}_{c_1} \times \mathcal{A}_{c_2}$ to (P') (with $c_2 > c_2'$) belongs in fact to $\mathcal{A}_{c_2'}$.*

*Proof.* Let $q \in \mathcal{Q}$. Using the convexity assumption on $\mathcal{G}$, we have

$$\begin{aligned}\mathcal{G}\left(q_T, X_T^{\bar{\psi}}\right) &\geq \mathcal{G}\left(q_T, 0\right) + \mathbb{E}\left[\delta_X \mathcal{G}(q_T, 0) \cdot X_T^{\bar{\psi}}\right] \\ &\geq -L - L\mathbb{E}\left[q_T |X_T^{\bar{\psi}}|\right],\end{aligned}$$

where the second inequality follows from the growth Assumption A6. Therefore,

$$\mathcal{J}(q, \bar{\psi}) + \mathcal{S}(q) \geq -L - L\mathbb{E}\left[q_T |X_T^{\bar{\psi}}|\right] + \mathbb{E}\left[\int_0^T q_t \ell(t, \bar{\psi}_t) \mathrm{d}t\right].$$

Since $\ell$ grows up at least quadratically fast by Assumption A4, we can find a constant $C > 0$ such that

$$\mathbb{E}\left[\int_0^T q_t \ell(t, \bar{\psi}_t) \mathrm{d}t\right] \geq -C + \frac{1}{2L}\mathbb{E}\left[\int_0^T q_t \left|\bar{\psi}_t\right|^2 \mathrm{d}t\right].$$

Moreover, we know from Lemma 41 that, for any $\varepsilon \in (0,1)$, there exists $C_\varepsilon > 0$ such that

$$\mathbb{E}\left[q_T |X_T^{\bar{\psi}}|\right] \leq C_\varepsilon + c_\varepsilon \mathcal{S}(q) + \varepsilon \mathbb{E}\left[\int_0^T q_t \left|\bar{\psi}_t\right|^2 \mathrm{d}t\right],$$

with $c_\varepsilon = 2\beta e^{\alpha T} \|\Gamma\|_{L^\infty(\mathbb{F}, \mathbb{R}^{n\times n})} \|\Gamma^{-1}\|_{L^\infty(\mathbb{F}, \mathbb{R}^{n\times n})} \left(\|\nu\|_{L^\infty(\mathbb{F}, \mathbb{R}^{n\times d})} + \frac{3}{\varepsilon} e^{\alpha T} \|\sigma\|_{L^\infty(\mathbb{F}, \mathbb{R}^{n\times d\times n})}\right)$.
Therefore, choosing $\varepsilon = 1/[4\max(1, L)]$ and combining the last three displays, we get

$$\mathcal{J}\left(q, \bar{\psi}\right) \geq -C - c_\varepsilon \mathcal{S}(q) + \frac{1}{4\max(1, L)} \mathbb{E}\left[\int_0^T q_t \left|\bar{\psi}_t\right|^2 \mathrm{d}t\right].$$

And, then for the same constants $\varepsilon$, $c_\varepsilon$ and $C$ as above,

$$\sup_{q \in \mathcal{Q}} \left\{\mathbb{E}\left[\int_0^T q_t \left|\bar{\psi}_t\right|^2 \mathrm{d}t\right] - \gamma \mathcal{S}(q)\right\} \leq 4\max(1, L)\left[C + \sup_{q \in \mathcal{Q}} \mathcal{J}\left(q, \bar{\psi}\right)\right]. \tag{192}$$

where we recall that $\gamma$, defined in Assumption A5, is given by

$$\begin{aligned}\gamma =& 4\max(1, L) e^{\alpha T} \|\Gamma\|_{L^\infty(\mathbb{F}, \mathbb{R}^{n\times n})} \|\Gamma^{-1}\|_{L^\infty(\mathbb{F}, \mathbb{R}^{n\times n})} \\ &\times \left(\|\nu\|_{L^\infty(\mathbb{F}, \mathbb{R}^{n\times d})} + 12\max(1, L) e^{\alpha T} \|\sigma\|_{L^\infty(\mathbb{F}, \mathbb{R}^{n\times d\times n})}\right).\end{aligned}$$

Recall now that $(\bar{q}, \bar{\psi})$ is a saddle point of (P'). By the third assertion in the statement of Theorem 25, we deduce that $\bar{q}$ is a maximizer of $q' \in \mathcal{Q} \mapsto \mathcal{J}(q', \bar{\psi})$. Therefore, the supremum in the above display can be bounded as follows

$$\sup_{q \in \mathcal{Q}} \mathcal{J}(q, \bar{\psi}) = \mathcal{J}(\bar{q}, \bar{\psi}) \leq \mathcal{J}(\bar{q}, 0), \tag{193}$$

with the last inequality following from the saddle point property of $(\bar{q}, \bar{\psi})$. Since $f^\star$ is lower bounded, we have

$$\mathcal{J}(\bar{q},0) = \mathcal{R}(\bar{q},0) - \mathcal{S}(\bar{q}) \leq C' \left(1 + \mathbb{E}\left[\bar{q}_T |X_T^0| + \int_0^T \bar{q}_t \ell(t,0)\mathrm{d}t\right]\right),$$

for a new constant $C'$. We then use (13) to upper bound the right-hand side. By Lemma 42, we know that there exists $\tau > 0$, only depending on the data such that $X_T^0 \in S^{1,\tau}_{\exp}(\mathcal{F}_T, \mathbb{R}^n)$. And then, (13) (together with the fact that $\bar{q} \in \mathcal{Q}_{c_1}$) yields

$$\mathcal{J}(\bar{q},0) \leq C'(1+c_1).$$

Returning to (192) and (193), we obtain

$$\sup_{q\in\mathcal{Q}} \left\{\mathbb{E}\left[\int_0^T q_t \left|\bar{\psi}_t\right|^2 \mathrm{d}t\right] - \gamma \mathcal{S}(q)\right\} \leq 4\max(1,L)\left[C + C'(1+c_1)\right].$$

This completes the proof. □

#### 5.3.2 Necessary conditions

This subsection is dedicated to a series of lemmas leading eventually to Lemma 37, which we invoked in the proof of Theorem 31 to establish the necessary condition. Given the constant $c_2'$ in Lemma 32, we assume that the constant $c_2$ in the definition of $\mathcal{A}_{c_2}$ in ($\mathrm{P_C}$) satisfies $c_2 > c_2' > 0$, which condition guarantees that any minimizer to ($\mathrm{P_C}$) –with $\bar{q} \in \mathcal{Q}_{c_1}$ such that, for some $\bar{\psi} \in \mathcal{A}_{c_2}$, $(\bar{q},\bar{\psi})$ is a saddle-point– lies 'in the interior' of the admissible set $\mathcal{A}_{c_2}$ (in the sense that it belongs to $\mathcal{A}_{c_2'}$). We insist on the fact that, similar to Subsubsection 5.3.1, $\bar{q}$ is fixed throughout the analysis. In coherence with the convention adopted earlier, this makes it possible to omit $\bar{q}$ in the various notations. For instance, we write $\delta_X \mathcal{G}(X_T)$ for $\delta_X \mathcal{G}(\bar{q}_T, X_T)$.

Recalling the definition of the pre-Hamiltonian $H$ in (20), and denoting $X^\psi$, for a given $\psi \in \mathcal{A}_{c_2'}$, the associated solution to the state equation, we define the adjoint BSDE with unknown $(p,k)$,

$$-\mathrm{d}p_t = \nabla_x H\left(t, X_t^\psi, \psi_t, p_t, k_t, \bar{q}_t\right)\mathrm{d}t - k_t \mathrm{d}W_t, \quad p_T = \delta_X \mathcal{G}(\bar{q}_T, X_T^\psi). \tag{194}$$

By assumptions on the mappings $b$ and $\sigma$, the derivative of the pre-Hamiltonian simplifies to

$$\nabla_x H\left(t, X_t^\psi, \psi_t, p_t, k_t, \bar{q}_t\right) = b_t^\top p_t.$$

In the analysis carried out below, we will also use the fact that

$$\nabla_\psi H(t,x,\psi,p,k,q) = q\nabla_\psi \ell(t,\psi) + c_t^\top p + r\mathrm{Tr}\left(\sigma_t^\top k\right), \tag{195}$$

where $r$ can be either 0 or 1, and we recall the convention (22) for the trace

$$\mathrm{Tr}\left(\sigma_t^\top k_t\right) = \left(\sum_{i=1}^n \sum_{j=1}^d (\sigma_t)_{i,j,\ell}(k_t)_{i,j}\right)_{\ell=1,\dots,d}. \tag{196}$$

To study the BSDE (194), we introduce the following intermediary BSDE with unknown $(\varrho, h)$,

$$-\mathrm{d}\varrho_t = \left(b_t^\top \varrho_t + \bar{Y}_t^\star \varrho_t\right) \mathrm{d}t - h_t \mathrm{d}\bar{W}_t, \quad \varrho_T = \bar{q}_T^{-1} \delta_X \mathcal{G}(\bar{q}_T, X_T^\psi), \tag{197}$$

where $(\bar{W}_t = W_t - \int_0^t \bar{Z}_s^\star \mathrm{d}s)_{t\in[0,T]}$ is a Brownian motion under the equivalent probability measure $\bar{\mathbb{Q}}$ defined by $\bar{\mathbb{Q}} = \bar{\mathcal{E}}_T \mathbb{P}$, with $\bar{\mathcal{E}}_T := \mathcal{E}_T(\int_0^\cdot \bar{Z}_s^\star \cdot \mathrm{d}W_s)$. We recall that $(\bar{Y}^\star, \bar{Z}^\star)$ is associated to $\bar{q}$ through (4), and that $\bar{q}_T = \exp(\int_0^T \bar{Y}_s^\star \mathrm{d}s)\bar{\mathcal{E}}_T$.

**Lemma 33.** *There exists a unique solution* $(\varrho, h) \in S^2(\mathbb{F}, \mathbb{R}^n, \bar{\mathbb{Q}}) \times M^2(\mathbb{F}, \mathbb{R}^{n\times d}, \bar{\mathbb{Q}})$ *to* (197)*. When* $r = 1$*,* $\varrho$ *belongs to* $L^\infty(\mathbb{F}, \mathbb{R}^n)$ *and* $h$ *to* $L^2(\mathbb{F}, \mathbb{R}^{n\times d}, \bar{\mathbb{Q}})$*.*

*Proof.* Let us recall that

$$\|b\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})} + \|\bar{Y}^\star\|_{L^\infty(\mathbb{F})} \leq \alpha + L, \tag{198}$$

by Assumption A1 and since $|\bar{Y}_s^\star| \leq \alpha$. In particular, the BSDE (197) is a linear BSDE, with a bounded linear coefficient. The existence and uniqueness of a solution are well established once the terminal condition has been shown to be sufficiently integrable.

*Step 1:* $r = 0$. By the growth Assumption A6 on $\delta_X \mathcal{G}$, we have

$$|\varrho_T| = \bar{q}_T^{-1} \left|\delta_X \mathcal{G}(\bar{q}_T, X_T^\psi)\right| \leq L\left(1 + |X_T^\psi| + \mathbb{E}\left[\bar{q}_T |X_T^\psi|^2\right]\right).$$

Taking the square on both sides, we get (for a constant $C$ only depending on $L$ and whose value is allowed to vary from line to line)

$$\begin{aligned}
\mathbb{E}^{\bar{\mathbb{Q}}}\left[|\varrho_T|^2\right] &\leq C\left(1 + \mathbb{E}^{\bar{\mathbb{Q}}}\left[|X_T^\psi|^2\right] + \mathbb{E}\left[\bar{q}_T |X_T^\psi|^2\right]^2\right) \\
&\leq C\left(1 + \mathbb{E}\left[\int_0^T \bar{q}_s |\psi_s|^2 \mathrm{d}s\right]^2\right) \\
&\leq C\left(1 + \mathcal{S}(\bar{q})^2 + \mathcal{S}^\star(\psi)^2\right) < +\infty,
\end{aligned}$$

where we used Lemma 40 in Appendix B in order to pass from the first to the second line. Together with (198), existence and uniqueness follow from the standard $L^2$-theory of BSDEs (we recall from [2, Theorem 2.4] that the martingale representation theorem holds under $\bar{\mathbb{Q}}$, with respect to $\bar{W}$).

*Step 2:* $r = 1$. In this case, $|\delta_X \mathcal{G}|$ is bounded by $C\bar{q}_T(1 + \mathbb{E}[\bar{q}_T|X_T|])$, for a constant $C$ only depending on $L$. By Lemma 40, $\mathbb{E}[\bar{q}_T|X_T|] < +\infty$. Therefore, the terminal condition in (197) is square integrable under the probability measure $\bar{\mathbb{Q}}$. The conclusion follows by the same arguments as in the first step. This concludes the proof. □

**Lemma 34.** *There exists a unique solution* $(p, k)$ *to* (194) *in the space* $D(\mathbb{F}, \mathbb{P}, \mathbb{R}^n) \times (\cap_{\beta\in(0,1)} M^\beta(\mathbb{F}, \mathbb{P}, \mathbb{R}^{n\times d}))$*. It is given by*

$$(p_t, k_t)_{t\in[0,T]} = \left(\bar{q}_t \varrho_t, \bar{q}_t(h_t + \varrho_t \otimes \bar{Z}_t^\star)\right)_{t\in[0,T]},$$

*where* $(\varrho_t, h_t)_{t\in[0,T]}$ *is the solution to* (197) *and* $(\varrho_t \otimes \bar{Z}_t^\star)_{t\in[0,T]}$ *denotes the* $n \times d$ *matrix with elements* $(\varrho_t^i \bar{Z}_t^{\star,j})_{i\in\{1,\dots,n\},j\in\{1,\dots,d\}}$*.*

*Proof. Step 1: Existence.* Repeating the computations from the proof of Lemma 33, there exists a constant $C$, only depending on $L$, such that

$$\begin{aligned}\mathbb{E}\left[\left|\delta_X\mathcal{G}\left(\bar{q}_T, X_T^\psi\right)\right|\right] &\le L\mathbb{E}\left[\bar{q}_T\left(1+\left|X_T^\psi\right|^{1-r}+\mathbb{E}\left[\bar{q}_T|X_T^\psi|^{2-r}\right)\right]\right]\\ &\le C\left(1+\mathbb{E}\left[\int_0^T \bar{q}_s|\psi_s|^2\mathrm{d}s\right]^2\right)\\ &\le C\left(1+\mathcal{S}(\bar{q})^2+\mathcal{S}^\star(\psi)^2\right)<+\infty.\end{aligned}$$

The existence of a solution to (194), within the space specified in the statement, follows from [23, Proposition 6.4].

*Step 2: Uniqueness.* Now let $(p_t,k_t)_{t\in[0,T]}$ be a solution to the backward equation (194) (within the space mentioned in the statement). Then, one can verify that $(\varrho_t,h_t)_{t\in[0,T]} := (\bar{q}_t^{-1}p_t, \bar{q}_t^{-1}(k_t - p_t\otimes\bar{Z}_t^\star))_{t\in[0,T]}$ solves (197), with the stochastic integral therein being understood as a local martingale under $\bar{\mathbb{Q}}$. That said, at this stage, we do not know yet that the hence defined pair $(\varrho,h)$ belongs to the space $S^2(\mathbb{F},\mathbb{R}^n,\bar{\mathbb{Q}})\times M^2(\mathbb{F},\mathbb{R}^{n\times d},\bar{\mathbb{Q}})$, which prevents us from identifying directly $(\varrho,h)$ with the (unique) solution constructed in Lemma 33. We thus proceed as follows. We denote by $(R_t)_{t\in[0,T]}$ the solution to the (random) ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}R_t = -b_t^\top R_t - \bar{Y}_t^\star R_t, \quad R_0 = I_n, \tag{199}$$

with $I_n$ denoting the $n\times n$ identity matrix. The process $(R_t)_{t\in[0,T]}$ takes values in the set of $n\times n$ invertible matrices, and

$$\frac{\mathrm{d}}{\mathrm{d}t}R_t^{-1} = R_t^{-1}b_t^\top + \bar{Y}_t^\star R_t^{-1}, \quad R_0^{-1} = I_n.$$

Obviously, the process $(R_t^{-1})_{t\in[0,T]}$ is bounded by a deterministic constant. Also, it is straightforward to check that $(R_t^{-1}\varrho_t)_{t\in[0,T]}$ is a local martingale under $\bar{\mathbb{Q}}$. By a standard localization argument, we can find a non-decreasing sequence of stopping times $(\tau_k)_{k\ge1}$, converging to $T$, such that, for any $t\in[0,T]$, and any integer $k\ge1$,

$$R_{t\wedge\tau_k}^{-1}\varrho_{t\wedge\tau_k} = \mathbb{E}^{\bar{\mathbb{Q}}}\left[R_{\tau_k}^{-1}\varrho_{\tau_k}|\mathcal{F}_t\right]. \tag{200}$$

In order to pass to the limit (as $k$ tends to $+\infty$) in the above display, we check that the collection of random variables $(R_{\tau_k}^{-1}\varrho_{\tau_k})_{k\ge1}$ is uniformly integrable under $\bar{\mathbb{Q}}$ (the same argument would show that the collection of random variables $(R_{t\wedge\tau_k}^{-1}\varrho_{t\wedge\tau_k})_{k\ge1}$ is uniformly integrable). For any event $A$, we can find a constant $C$ such that, for any $k\ge1$,

$$\mathbb{E}^{\bar{\mathbb{Q}}}\left[\mathbb{1}_A|R_{\tau_k}^{-1}\varrho_{\tau_k}|\right] \le C\mathbb{E}\left[\mathbb{1}_A|q_{\tau_k}\varrho_{\tau_k}|\right] = C\mathbb{E}\left[\mathbb{1}_A|p_{\tau_k}|\right].$$

Since $p$ belongs to $D(\mathbb{F},\mathbb{P})$, the right-hand side tends to 0, uniformly in $k$, as $\mathbb{P}(A)$ tends to 0. Writing $\mathbb{P}(A) = \mathbb{E}^{\bar{\mathbb{Q}}}[(\bar{\mathcal{E}}_T)^{-1}\mathbb{1}_A]$ (here $\bar{\mathcal{E}}_T>0$ a.s., because $\bar{q}_T>0$ a.s.), we deduce that $\mathbb{P}(A)$ tends to 0 as $\bar{\mathbb{Q}}(A)$ tends to 0. This shows that the right-hand side (in the above display) tends to 0, uniformly in $k$, as $\bar{\mathbb{Q}}(A)$ tends to 0, which provides the required uniform integrability property. Letting $k$ tend to $+\infty$ in (200), we deduce that, for any $t\in[0,T]$,

$$R_t^{-1}\varrho_t = \mathbb{E}^{\bar{\mathbb{Q}}}\left[R_T^{-1}\bar{q}_T^{-1}\delta_X\mathcal{G}(\bar{q}_T,X_T^\psi)\middle|\mathcal{F}_t\right].$$

This provides an explicit formula for $\varrho$ and makes it possible to identify it (together with $h$) with the solution obtained in Lemma 33. It remains to see that the mapping

$$\mathbb{R}^n \times \mathbb{R}^{n\times d} \ni (p,k) \mapsto (q^{-1}p, q^{-1}(k - p \otimes z^\star)),$$

is one-to-one for any $q > 0$ and $z^\star \in \mathbb{R}^n$. This proves that $(p_t,k_t)_{t\in[0,T]}$ is uniquely determined by the pair $(\varrho_t,h_t)_{t\in[0,T]}$ and is thus unique. □

**Tangent processes.** Let $\varphi \in L^\infty(\mathbb{F},\mathbb{R}^n)$ be such that $\psi+\varphi \in \mathcal{A}_{c_2}$ (we recall that $\psi$ is an arbitrary element in $\mathcal{A}_{c_2'}$). We introduce the following system of variational (or tangent) processes with unknown $(u,v,x)$, the latter taking values in $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^n$,

$$\begin{cases} \mathrm{d}u_t &= \bar{q}_t\varphi_t \cdot \nabla_\psi \ell(t,\psi_t)\mathrm{d}t - v_t \cdot \mathrm{d}W_t, \quad u_T = \delta_X \mathcal{G}(\bar{q}_T, X_T^\psi)\cdot x_T, \\ \mathrm{d}x_t &= (b_t x_t + c_t\varphi_t)\,\mathrm{d}t + rD_\psi\sigma_t(\varphi_t)\mathrm{d}W_t, \quad x_0 = 0, \end{cases} \tag{V}$$

where we recall that $D_\psi\sigma_t(\varphi_t) = (\sum_{\ell=1}^n (\sigma_t)_{i,j,\ell}(\varphi_t)_\ell)_{i\in\{1,\dots,n\},j\in\{1,\dots,d\}}$, see A2.

**Lemma 35.** *Let $\varphi \in L^\infty(\mathbb{F},\mathbb{R}^n)$ such that $\psi+\varphi\in\mathcal{A}_{c_2}$. Then, there exists a unique solution $(x_t)_{t\in[0,T]}$ to the forward equation in (V), lying in $S^2(\mathbb{F},\mathbb{R}^n)\cap S^2(\mathbb{F},\mathbb{R}^n,\bar{\mathbb{Q}})$ (even in $L^\infty(\mathbb{F},\mathbb{R}^n)$ when $r=0$). And, there exists a solution $(u,v)\in D(\mathbb{F},\mathbb{P})\times(\cap_{\beta\in(0,1)}M^\beta(\mathbb{F},\mathbb{R}^d,\mathbb{P}))$ to the backward equation in (V).*

*Proof. Step 1: Existence and uniqueness to the forward equation.* Denoting by $\Gamma$ the resolvent associated with the linear part of the forward equation, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_t = b_t\Gamma_t, \quad \Gamma_0 = I_n,$$

with $I_n$ standing for the $n\times n$ identity matrix, the solution to the forward equation in (V) is explicitly given by

$$x_t = \Gamma_t\left(\eta + \int_0^t \Gamma_s^{-1}(a_s + c_s\varphi_s)\,\mathrm{d}s + r\int_0^t \Gamma_s^{-1}D_\psi\sigma_s(\varphi_s)\mathrm{d}W_s\right), \quad t\in[0,T].$$

Thus, $(x_t)_{t\in[0,T]}$ clearly belongs to $S^2(\mathbb{F},\mathbb{R}^n,\mathbb{P})$ (because $\varphi$ is bounded). It further belongs to $S^2(\mathbb{F},\mathbb{R}^n,\bar{\mathbb{Q}})$ by a direct application of Lemma 40 using once again the fact that $\varphi$ is bounded (where we identify the process $(\Gamma_t^{-1}D_\psi\sigma_t(\varphi_t))_{t\in[0,T]}$, which is bounded, in this proof with the process $(\nu_t)_{t\in[0,T]}$ in the statement of Lemma 40).

*Step 2: Well-posedness of the backward equation.* By the growth assumptions A4 and A6, we have

$$\mathbb{E}\left[|\delta_X\mathcal{G}(\bar{q}_T,X_T^\psi)\cdot x_T|\right] \le C\mathbb{E}\left[\bar{q}_T(1+|X_T^\psi|^{1-r})|x_T|\right].$$

When $r=1$, the right-hand side reduces to $C(1+\mathbb{E}[\bar{q}_T|x_T|])$, which is finite since $(x_t)_{t\in[0,T]}\in S^2(\mathbb{F},\mathbb{R}^n,\bar{\mathbb{Q}})$. When $r=0$, $(x_t)_{t\in[0,T]}\in L^\infty(\mathbb{F},\mathbb{R}^n,\mathbb{P})$, and by Lemma 40, we have

$$\mathbb{E}\left[|\delta_X\mathcal{G}(\bar{q}_T,X_T^\psi)\cdot x_T|\right] \le C\left(1+\mathbb{E}\left[\bar{q}_T|X_T^\psi|\right]\right) < +\infty.$$

Since $\nabla_\psi \ell$ is at most of linear growth in $\psi$, we further have

$$\begin{aligned}\mathbb{E}\left[\int_0^T \bar{q}_s|\varphi_s \cdot \nabla_\psi \ell(s,\psi_s)|\mathrm{d}s\right] &\leq C\mathbb{E}\left[\int_0^T \bar{q}_s\left(1+|\varphi_s|^2+|\psi_s|^2\right)\mathrm{d}s\right] \\ &\leq C\left(1+\mathcal{S}(\bar{q})+\mathcal{S}^\star(\psi)\right) < +\infty.\end{aligned} \tag{201}$$

The conclusion follows by [23, Proposition 6.4]. □

So far, we have considered $\varphi \in L^\infty(\mathbb{F},\mathbb{R}^n,\mathbb{P})$ such that $\psi+\varphi \in \mathcal{A}_{c_2}$. Since $\mathcal{A}$ is convex, we then have, for any $\varepsilon \in [0,1]$, $\psi+\varepsilon\varphi \in \mathcal{A}_{c_2}$. By optimality of the control $\psi$, we get

$$\varepsilon^{-1}\left(\mathcal{J}(\psi+\varepsilon\varphi)-\mathcal{J}(\psi)\right) \geq 0. \tag{202}$$

We use the above inequality to prove the following statement:

**Lemma 36.** *Let $\varphi \in L^\infty(\mathbb{F},\mathbb{R}^n,\mathbb{P})$ such that $\psi+\varphi \in \mathcal{A}_{c_2}$. With $(u,v)$ being as in the statement of Lemma 35, the following variational inequality holds true:*

$$\mathbb{E}[u_0] = \mathbb{E}\left[u_T + \int_0^T \bar{q}_s\varphi_s \cdot \nabla_\psi \ell(s,\psi_s)\mathrm{d}s\right] \geq 0.$$

*Proof.* Since $\bar{q}$ is fixed, we omit to indicate it explicitly in the various functionals that depend on it. For instance, we use the shorthand notation $\mathcal{G}(X_T^\psi)$ for $\mathcal{G}(\bar{q}_T, X_T^\psi)$.

On the one hand, we have, from A7,

$$\mathcal{G}(X_T^{\psi+\varepsilon\varphi})-\mathcal{G}(X_T^\psi) = \mathbb{E}\left[\delta_X\mathcal{G}(X_T^\psi)\cdot\left(X_T^{\psi+\varepsilon\varphi}-X_T^\psi\right)\right] + O\left(\mathbb{E}\left[\bar{q}_T|X_T^{\psi+\varepsilon\varphi}-X_T^\psi|^2\right]\right).$$

Because $(X_t)_{t\in[0,T]}$ solves a linear SDE, we also have

$$\varepsilon^{-1}(X_T^{\psi+\varepsilon\varphi}-X_T^\psi) = X_T^\varphi = x_T,$$

and thus

$$\varepsilon^{-1}\left(\mathcal{G}(X_T^{\psi+\varepsilon\varphi})-\mathcal{G}(X_T^\psi)\right) = \mathbb{E}\left[\delta_X\mathcal{G}(X_T^\psi)\cdot x_T\right] + \varepsilon O\left(\mathbb{E}\left[\bar{q}_T|x_T|^2\right]\right). \tag{203}$$

In order to handle the right-hand side, we use the same estimates as in the second step of the proof of Lemma 35. In particular, we already know that $\mathbb{E}[|\delta_X\mathcal{G}(X_T^\psi)\cdot x_T|] < +\infty$. We also know that $\mathbb{E}[\bar{q}_T|x_T|^2] < +\infty$, from which we deduce that the term on the second line of (203) tends to 0 with $\varepsilon$. So, we obtain

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\left(\mathcal{G}(X_T^{\psi+\varepsilon\varphi})-\mathcal{G}(X_T^\psi)\right) = \mathbb{E}\left[\delta_X\mathcal{G}(X_T^\psi)\cdot x_T\right]. \tag{204}$$

Since $\ell$ is assumed to be twice differentiable, with bounded second-order derivatives, we also have

$$\begin{aligned}\varepsilon^{-1}\mathbb{E}\left[\int_0^T \bar{q}_s\left(\ell(s,\psi_s+\varepsilon\varphi_s)-\ell(s,\psi_s)\right)\mathrm{d}s\right] =& \mathbb{E}\left[\int_0^T \bar{q}_s\varphi_s\cdot\nabla_\psi\ell(s,\psi_s)\mathrm{d}s\right] \\ &+ \varepsilon o\left(\mathbb{E}\left[\int_0^T \bar{q}_s|\varphi_s|^2\mathrm{d}s\right]\right).\end{aligned}$$

Because $\varphi \in L^\infty(\mathbb{F},\mathbb{R}^n)$, we obviously have $\mathbb{E}[\int_0^T \bar{q}_s|\varphi_s|^2\mathrm{d}s] < +\infty$. As for the first term on the right-hand side, we recall from (201) that it is bounded. Then, combining the above display with (204), and recalling again the definition of the criterion $\mathcal{J}$, we get

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\left(\mathcal{J}(\psi+\varepsilon\varphi)-\mathcal{J}(\psi)\right) = \mathbb{E}\left[\delta_X \mathcal{G}(X_T^{\psi})\cdot x_T\right] + \mathbb{E}\left[\int_0^T \bar{q}_s\varphi_s\cdot\nabla_\psi \ell(s,\psi_s)\mathrm{d}s\right]$$
$$= \mathbb{E}\left[u_T + \int_0^T \bar{q}_s\varphi_s\cdot\nabla_\psi \ell(s,\psi_s)\mathrm{d}s\right].$$

It remains to use to the backward equation in (V) in order to identify the last term with $\mathbb{E}[u_0]$. By localization, we can find a non-decreasing sequence of stopping times $(\tau_m)_{m\geq 1}$, converging to $T$, such that, for any $m \geq 1$,

$$\mathbb{E}\left[u_{\tau_m} + \int_0^{\tau_m} \bar{q}_s\varphi_s\cdot\nabla_\psi \ell(s,\psi_s)\mathrm{d}s\right] = \mathbb{E}[u_0]. \tag{205}$$

Since $u$ belongs to $D(\mathbb{F},\mathbb{P})$, $\mathbb{E}[u_{\tau_m}] \to \mathbb{E}[u_T]$ as $m$ tends to $+\infty$. And by (201), $\mathbb{E}[\int_0^{\tau_m} \bar{q}_s\varphi_s\cdot\nabla_\psi \ell(s,\psi)\mathrm{d}s] \to \mathbb{E}[\int_0^T \bar{q}_s\varphi_s\cdot\nabla_\psi \ell(s,\psi)\mathrm{d}s]$. This shows that the left-hand side in the above display converges to the right-hand side of (205). Recalling inequality (202), we complete the proof. □

**Lemma 37.** *Let $\psi \in \mathcal{A}_{c_2'}$ be a minimizer to problem ($\mathrm{P_C}$) and $(p,k)$ the associated solution to (194) provided by Lemma 34. Then, $(\psi,p,k,X)$ belongs to the set $\mathscr{A}$ defined in (25) and satisfies the first-order condition ($\mathrm{Opt_C}$) (with $\bar{q}$ being substituted for $q$ therein).*

*Proof.* We start with the following preliminary remark: the fact that $(\psi,p,k,X)$ belongs to $\mathscr{A}$ is a consequence of Lemmas 34 and 40.

Next, following the analysis carried out in Lemmas 35 and 36, we consider $\varphi \in L^\infty(\mathbb{F},\mathbb{R}^n,\mathbb{P})$ such that $\psi+\varphi \in \mathcal{A}_{c_2}$. With $x$ as in Lemma 35, and by Itô's formula, we have

$$p_T\cdot x_T = \int_0^T p_s\cdot \mathrm{d}x_s + \int_0^T x_s\cdot \mathrm{d}p_s + r\int_0^T \mathrm{Tr}((D_\psi\sigma_s(\varphi_s))^\top k_s)\mathrm{d}s.$$

For a given $A>0$, we also consider the stopping time

$$\tau_A := \inf\left\{t\in[0,T],\ \left|\int_0^t p_s^\top D_\psi\sigma_s(\varphi_s)\mathrm{d}W_s\right| + \left|\int_0^t x_s^\top k_s \mathrm{d}W_s\right| \geq A\right\}.$$

Then, by cancellation of the expectations of the stopped stochastic integrals in the expansion of $(p_t\cdot x_t)_{t\in[0,T]}$, we get

$$\mathbb{E}\left[p_{T\wedge\tau_A}\cdot x_{T\wedge\tau_A}\right] = \mathbb{E}\left[\int_0^{T\wedge\tau_A} p_s^\top c_s\varphi_s\mathrm{d}s + r\int_0^{T\wedge\tau_A}\mathrm{Tr}((D_\psi\sigma_s(\varphi_s))^\top k_s)\mathrm{d}s\right]. \tag{206}$$

We now aim to pass to the limit on both sides of the equality.

*Step 1: convergence of the left-hand side of (206).* To pass to the limit, we establish that the random variables $(p_{T\wedge\tau_A}\cdot x_{T\wedge\tau_A})_{A>0}$ are uniformly integrable, distinguishing between the two cases $r=0$ and $r=1$. Throughout, $E$ is a fixed subset of $\Omega$, belonging to $\mathcal{F}_T$.

When $r = 0$, the process $x = (x_t)_{t\in[0,T]}$ is bounded (see Lemma 35). Then, we can find a constant $C$, independent of $A$, such that

$$\mathbb{E}\left[\mathbb{1}_E |p_{T\wedge\tau_A} \cdot x_{T\wedge\tau_A}|\right] \le C\mathbb{E}\left[\mathbb{1}_E |p_{T\wedge\tau_A}|\right].$$

Since $p$ belongs to $D(\mathbb{F}, \mathbb{R}^n, \mathbb{P})$ by Lemma 34, the right-hand side tends to 0, uniformly with respect to $A$, as $\mathbb{P}(E)$ tends to 0, yielding the required uniform integrability property.

When $r = 1$, the process $x$ belongs to $S^2(\mathbb{F}, \mathbb{R}^n, \bar{\mathbb{Q}})$ thanks to Lemma 35. By Lemmas 33 and 34, the process $p$ is equal to $\bar{q}\varrho$ with $\varrho$ belonging to $L^\infty(\mathbb{F}, \mathbb{R}^n)$. Then, we have

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_E |p_{T\wedge\tau_A} \cdot x_{T\wedge\tau_A}|\right] &= C\mathbb{E}\left[\mathbb{1}_E \bar{q}_{T\wedge\tau_A} |\varrho_{T\wedge\tau_A} \cdot x_{T\wedge\tau_A}|\right] \\
&\le C\mathbb{E}\left[\mathbb{1}_E \bar{q}_T^* |x_{T\wedge\tau_A}|\right] \\
&\le C\mathbb{E}\left[\bar{q}_T^* \mathbb{1}_E\right]^{1/2},
\end{aligned}$$

with the last line following from Cauchy-Schwarz inequality, and, once again, from the fact that $x \in S^2(\mathbb{F}, \mathbb{R}^n, \bar{\mathbb{Q}})$. In particular, the constant $C$ on the last line is allowed to depend on the $S^2$-norm of $x$ (under $\bar{\mathbb{Q}}$) and is implicitly allowed to vary from line to line. To prove that the term on the last line of the above displays tends to 0 as $\mathbb{P}(E)$ tends to 0, it suffices to recall from Lemma 39 that $q_T^*$ is integrable, so that the right-hand side tends to 0 as $\mathbb{P}(E)$ tends to 0. This yields the expected uniform integrability property.

*Step 2: convergence on the right-hand side of* (206). Using the fact that the terms $c, \varphi$ and $D_\psi\sigma(\varphi_s)$ are uniformly bounded, and that $(p, k)$ belongs to $D(\mathbb{F}, \mathbb{R}^n, \mathbb{P}) \times (\cap_{\beta\in(0,1)} M^\beta(\mathbb{F}, \mathbb{R}^{n\times d}, \mathbb{P}))$, see Lemma 34, we can derive the following upper-bound

$$\begin{aligned}
&\int_0^{T\wedge\tau_A} \left| p_s^\top c_s \varphi_s \right| \mathrm{d}s + r \int_0^{T\wedge\tau_A} \left| \mathrm{Tr}((D_\psi\sigma_s(\varphi_s))^\top k_s) \right| \mathrm{d}s \\
&\le C \left( \int_0^T |p_s| \mathrm{d}s + r \int_0^T |k_s| \mathrm{d}s \right).
\end{aligned}$$

When $r = 0$, we deduce that the left-hand side is integrable, uniformly with respect to $A$. When $r = 1$, the proof is more involved. We recall from Lemmas 33 and 34 that $|p_t| \le C\bar{q}_t$ and $|k_t| \le \bar{q}_t(|h_t| + |\bar{Z}_t^\star|)$, for all $t \in [0, T]$, where $h \in L^2(\mathbb{F}, \mathbb{R}^{n\times d}, \bar{\mathbb{Q}})$. Since $\bar{Z}^\star$ belongs to $L^2(\mathbb{F}, \mathbb{R}^d, \bar{\mathbb{Q}})$, we deduce from Cauchy-Schwarz inequality that $k$ belongs to $L^1(\mathbb{F}, \mathbb{R}^{n\times d}, \mathbb{P})$. Therefore, the left-hand side in the above display is also integrable, uniformly with respect to $A > 0$.

*Step 3: conclusion.* Thanks to the uniform integrability properties established in the last two steps, we can now pass to the limit in (206). We get

$$\mathbb{E}\left[p_T \cdot x_T\right] = \mathbb{E}\left[\int_0^T p_s^\top c_s \varphi_s \mathrm{d}s + r \int_0^T \mathrm{Tr}((D_\psi\sigma_s(\varphi_s))^\top k_s) \mathrm{d}s\right].$$

Consider now $(u, v)$ as in the statement of Lemma 36. Using the fact that $u_T = p_T \cdot x_T$

and $\mathbb{E}[u_0] \geq 0$, together with Lemma 36, we obtain

$$\begin{aligned} 0 &\leq \mathbb{E}[u_0] \\ &= \mathbb{E}\left[p_T \cdot x_T + \int_0^T \varphi_s \cdot \bar{q}_s \nabla_\psi \ell(s, \psi_s) \mathrm{d}s\right] \\ &= \mathbb{E}\left[\int_0^T \left\{\varphi_s \cdot \left(\bar{q}_s \nabla_\psi \ell(s, \psi_s) + c_s^\top p_s\right) + r \mathrm{Tr}((D_\psi \sigma_s(\varphi_s)))^\top k_s)\right\} \mathrm{d}s\right] \\ &= \mathbb{E}\left[\int_0^T \varphi_s \cdot \nabla_\psi H(s, X_s, \psi_s, p_s, k_s, \bar{q}_s) \mathrm{d}s\right], \end{aligned} \tag{207}$$

where, to get the last line, we used (195) together with the identity

$$\mathrm{Tr}\left((D_\psi \sigma_s)^\top k_s)\right) = \sum_{i=1}^n \sum_{j=1}^d \sum_{\ell=1}^n (\sigma_s)_{i,j,k} (\varphi_s)_k (k_s)_{j,i} = \varphi_s \cdot \left(\sigma_s^\top k_s\right).$$

The sequence of inequalities (207) is true for any arbitrary perturbation $\varphi \in L^\infty(\mathbb{F}, \mathbb{R}^n)$ such that $\psi + \varphi \in \mathcal{A}_{c_2}$. In fact, using the property that $\psi \in \mathcal{A}_{c_2'}$, where $c_2' < c_2$, it is easy to see that, for any given $\varphi \in L^\infty(\mathbb{F}, \mathbb{R}^n)$, there exists an $\varepsilon_0$ (depending on $\varphi$) such that $\psi + \varepsilon_0 \varphi \in \mathcal{A}_{c_2}$. Substituting $\varepsilon_0 \varphi$ for $\varphi$ in (207), this proves that, for any $\varphi \in L^\infty(\mathbb{F}, \mathbb{R}^n)$,

$$\mathbb{E}\left[\int_0^T \varphi_s \cdot \nabla_\psi H(s, X_s, \psi_s, p_s, k_s, \bar{q}_s) \mathrm{d}s\right] \geq 0.$$

And then, changing $\varphi$ into $-\varphi$, the above inequality is in fact an equality, from which we get that

$$\nabla_\psi H(t, X_t, \psi_t, p_t, k_t, \bar{q}_t) = 0, \quad \mathrm{d}\mathbb{P} \otimes \mathrm{d}t\text{-a.e.}\ .$$

Since $H$ is strictly convex in the variable $\psi$, this shows that the third line in ($\mathrm{Opt}_\mathrm{C}$) is satisfied and concludes the proof. □

### 5.3.3 Sufficient conditions

This last subsection is dedicated to the proof of the following lemma, which we invoked in the proof of Theorem 31 to establish the sufficient condition.

**Lemma 38.** *Let $(\psi, p, k, X) \in \mathscr{A}$ be a solution to the first order condition* ($\mathrm{Opt}_\mathrm{C}$). *Then, $\psi$ is the (unique) minimizer of the problem* ($\mathrm{P}_\mathrm{C}$) *(with $\bar{q}$ being substituted for $q$ therein).*

*Proof.* Let $\psi' \in \mathcal{A}_{c_2}$ and $X' := X^{\psi'}$ be the associated state. By definition of $\mathcal{J}$ (which is here a shorthand notation for $\mathcal{J}(\bar{q}, \cdot)$), we have (with a similar shorthand notation for $\mathcal{G}(\bar{q}_T, \cdot)$)

$$\mathcal{J}(\psi') - \mathcal{J}(\psi) = \mathcal{G}(X_T') - \mathcal{G}(X_T) + \mathbb{E}\left[\int_0^T \bar{q}_s(\ell(s, \psi_s') - \ell(s, \psi_s)) \mathrm{d}s\right]. \tag{208}$$

By Assumption A8, the mapping $\mathcal{G}$ is convex with respect to its last variable. Therefore,

$$\mathcal{G}(X_T') - \mathcal{G}(X_T) \geq \mathbb{E}\left[\delta_X \mathcal{G}(X_T) \cdot (X_T' - X_T)\right] = \mathbb{E}\left[p_T \cdot (X_T' - X_T)\right]. \tag{209}$$

For a given $A > 0$, consider the stopping time

$$\tau_A := \inf\left\{t \in [0,T],\ \left|\int_0^t p_s^\top(\sigma_s(\psi_s') - \sigma_s(\psi_s))\mathrm{d}W_s\right| + \left|\int_0^t (X_s' - X_s)^\top k_s \mathrm{d}W_s\right| \geq A\right\}.$$

By (194) and Itô's formula, we have the following formula (which is the analogue of (206))

$$\begin{aligned}&\mathbb{E}\left[p_{T\wedge\tau_A}\cdot(X'_{T\wedge\tau_A} - X_{T\wedge\tau_A})\right]\\ &= \mathbb{E}\left[\int_0^{T\wedge\tau_A}\left(p_s^\top c_s(\psi_s' - \psi_s) + \mathrm{Tr}\left(k_s\left(\sigma(s,\psi_s') - \sigma(s,\psi_s)\right)^T\right)\right)\mathrm{d}s\right].\end{aligned} \tag{210}$$

Following the proof of Lemma 37, we now aim to take the limit $A \to +\infty$ (but the proof is more difficult because the difference $\psi' - \psi$, which is the analogue of $\varphi$ in the proof of Lemma 37, is not bounded). The strategy is to prove that the random variables inside the expectations are uniformly integrable with respect to $A$.

*Step 1: left-hand side of* (210). In this step, we check that the left-hand side on (210) is uniformly integrable with respect to $A$. Throughout, we consider a fixed event $E \in \mathcal{F}_T$. By Lemma 33, the process $p$ is equal to the process $\bar{q}\varrho$. Therefore,

$$\mathbb{E}\left[\mathbb{1}_E|p_{T\wedge\tau_A}\cdot(X'_{T\wedge\tau_A} - X_{T\wedge\tau_A})|\right] = C\mathbb{E}\left[\mathbb{1}_E\bar{q}_{T\wedge\tau_A}|\varrho_{T\wedge\tau_A}\cdot(X'_{T\wedge\tau_A} - X_{T\wedge\tau_A})|\right].$$

We recall that $X, X' \in S^{2-r}(\mathbb{F},\mathbb{R}^n,\bar{\mathbb{Q}})$, see Lemma 40. In addition, by Lemma 33, the process $\varrho$ belongs to $S^2(\mathbb{F},\mathbb{R}^n,\bar{\mathbb{Q}})$ when $r = 0$ and $L^\infty(\mathbb{F},\mathbb{R}^n)$ when $r = 1$.

Let us first study the case $r = 0$. By Young's inequality, we have, for any $\varepsilon > 0$,

$$\begin{aligned}&\mathbb{E}\left[\mathbb{1}_E\bar{q}_{T\wedge\tau_A}|\varrho_{T\wedge\tau_A}\cdot(X'_{T\wedge\tau_A} - X_{T\wedge\tau_A})|\right]\\ &\leq \varepsilon^{-1}\mathbb{E}\left[\mathbb{1}_E\bar{q}_{T\wedge\tau_A}|\varrho_{T\wedge\tau_A}|^2\right] + \varepsilon\mathbb{E}\left[\bar{q}_{T\wedge\tau_A}|X_{T\wedge\tau_A} - X'_{T\wedge\tau_A}|^2\right]\\ &\leq C\varepsilon^{-1}\mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_{T\wedge\tau_A})\bar{q}_T|\varrho_T^*|^2\right] + C\varepsilon\mathbb{E}\left[\bar{q}_T|X_T^*|^2\right] + C\varepsilon\mathbb{E}\left[\bar{q}_T|(X')_T^*|^2\right].\end{aligned}$$

Observing from Doob's inequality that $\sup_{t\in[0,T]}\mathbb{P}(E|\mathcal{F}_t)$ tends to 0 in $L^1(\mathbb{P})$ as $\mathbb{P}(E)$ tends to 0, we deduce that the left-hand side tends to 0 with $\mathbb{P}(E)$, uniformly in $A$. This proves the expected uniform integrability property when $r = 0$.

When $r = 1$, we have

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_E\bar{q}_{T\wedge\tau_A}|\varrho_{T\wedge\tau_A}\cdot(X'_{T\wedge\tau_A} - X_{T\wedge\tau_A})|\right] &\leq C\mathbb{E}\left[\mathbb{1}_E\bar{q}_{T\wedge\tau_A}|X'_{T\wedge\tau_A} - X_{T\wedge\tau_A})|\right]\\ &\leq C\mathbb{E}\left[\mathbb{P}(E|\mathcal{F}_{T\wedge\tau_A})\bar{q}_T|(X - X')_T^*|\right],\end{aligned}$$

and we conclude as in the case $r = 0$. We deduce that

$$\lim_{A\to+\infty}\mathbb{E}\left[p_{T\wedge\tau_A}\cdot(X'_{T\wedge\tau_A} - X_{T\wedge\tau_A})\right] = \mathbb{E}\left[p_T\cdot(X'_T - X_T)\right]. \tag{211}$$

*Step 2: right-hand side of* (210). Consider now the term on the right-hand side of (210). Obviously,

$$\begin{aligned}&\int_0^{T\wedge\tau_A}\left|p_s^\top c_s(\psi_s' - \psi_s) + \mathrm{Tr}\left(k_s\left(\sigma(s,\psi_s') - \sigma(s,\psi_s)\right)^T\right)\right|\mathrm{d}s\\ &\leq \int_0^T\left|q_s\varrho_s^\top c_s(\psi_s' - \psi_s) + q_s\mathrm{Tr}\left((h_s + \varrho_s\otimes\bar{Z}_s^\star)\left(\sigma(s,\psi_s') - \sigma(s,\psi_s)\right)^T\right)\right|\mathrm{d}s.\end{aligned}$$

For our purpose, it suffices to prove the integrability of the right-hand side. When $r = 0$, the volatility term is independent of the control and reduces to $\sigma(t, \psi_t) = \nu_t$, see Assumption A2 . Moreover, $(\varrho, h) \in S^2(\mathbb{F}, \mathbb{R}^n, \bar{\mathbb{Q}}) \times M^2(\mathbb{F}, \mathbb{R}^{n\times d}, \bar{\mathbb{Q}})$. Then, by Cauchy-Schwarz and Young inequalities,

$$\mathbb{E}\left[\int_0^T \bar{q}_s \left|\varrho_s c_s(\psi'_s - \psi_s)\right| \mathrm{d}s\right] \leq C\mathbb{E}\left[\int_0^T \bar{q}_s(|\varrho_s|^2 + |\psi'_s|^2 + |\psi_s|^2)\mathrm{d}s\right]^{1/2} < +\infty,$$

and

$$\mathbb{E}\left[\int_0^T \bar{q}_s \left|\mathrm{Tr}\left((h_s + \varrho_s \otimes \bar{Z}^\star_s)\left(\sigma(s, \psi'_s) - \sigma(s, \psi_s)\right)^T\right)\right| \mathrm{d}s\right] = 0.$$

since $\sigma(s, \psi_s) = \sigma(s, \psi'_s) = \nu_s$ almost surely, for all $s \in [0, T]$. Combining the last two displays yields

$$\mathbb{E}\left[\int_0^T \left|p_s^\top c_s(\psi'_s - \psi_s) + \mathrm{Tr}\left(k_s\left(\sigma(s, \psi'_s) - \sigma(s, \psi_s)\right)^T\right)\right| \mathrm{d}s\right] < +\infty. \tag{212}$$

Now, when $r = 1$, the volatility term $\sigma(s, \psi'_s) - \sigma(s, \psi_s)$ is no longer null and depends linearly on the difference $\psi'_s - \psi_s$. Moreover, the process $\varrho$ is bounded, see Lemma 33. Therefore,

$$\mathbb{E}\left[\int_0^T \bar{q}_s \left|\varrho_s c_s(\psi'_s - \psi_s)\mathrm{d}s\right|\right] \leq C\mathbb{E}\left[\int_0^T \bar{q}_s(|\psi'_s| + |\psi_s|)\mathrm{d}s\right] < +\infty,$$

and, by Cauchy-Schwarz and Young inequalities, we further have

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \bar{q}_s \left|\mathrm{Tr}\left((h_s + \varrho_s \otimes \bar{Z}^\star_s)\left(\sigma(s, \psi'_s) - \sigma(s, \psi_s)\right)^T\right)\right| \mathrm{d}s\right]\\ &\leq C\mathbb{E}\left[\int_0^T \bar{q}_s(|h_s|^2 + |Z^\star_s|^2 + |\psi'_s|^2 + |\psi_s|^2)\mathrm{d}s\right] < +\infty,\end{aligned}$$

so that (212) is also true when $r = 1$. Thus, by dominated convergence theorem, we have, for $r \in \{0, 1\}$,

$$\begin{aligned}&\lim_{A\to+\infty} \mathbb{E}\left[\int_0^{T\wedge\tau_A} \left(p_s^\top c_s(\psi'_s - \psi_s) + \mathrm{Tr}\left(k_s\left(\sigma(s, \psi'_s) - \sigma(s, \psi_s)\right)^T\right)\right) \mathrm{d}s\right]\\ &= \mathbb{E}\left[\int_0^T \left(p_s^\top c_s(\psi'_s - \psi_s) + \mathrm{Tr}\left(k_s\left(\sigma(s, \psi'_s) - \sigma(s, \psi_s)\right)^T\right)\right) \mathrm{d}s\right].\end{aligned} \tag{213}$$

*Step 3: conclusion.* By (211) and (213), we can pass to the limit in (210), letting $A$ tend to $+\infty$ therein. Informally, this means that we can substitute $T$ for $T \wedge \tau_A$ in (210), from which we get

$$\mathbb{E}\left[p_T \cdot (X'_T - X_T)\right] = \mathbb{E}\left[\int_0^T \left(p_s^\top c_s(\psi'_s - \psi_s) + \mathrm{Tr}\left(k_s\left(\sigma(s, \psi'_s) - \sigma(s, \psi_s)\right)^T\right)\right) \mathrm{d}s\right].$$

Finally, using that $\nabla_\psi H(t, X_t, \psi_t, p_t, k_t, \bar{q}_t) = 0$ (see (195)), we have $\bar{q}_t \nabla_\psi \ell(t, \psi_t) + c_t^\top p_t + r\mathrm{Tr}(\sigma_t^\top k_t) = 0$. Combining the above display with (208) and (209), we get

$$\mathcal{J}(\psi') - \mathcal{J}(\psi) \geq \mathbb{E}\left[\int_0^T \bar{q}_s\left(\ell(s, \psi'_s) - \ell(s, \psi_s) - \nabla_\psi \ell(s, \psi_s) \cdot (\psi'_s - \psi_s)\right) \mathrm{d}s\right] \geq 0,$$

where, by strict convexity of $\ell$ and strict positivity of $\bar{q}$, the last inequality is strict whenever $\psi \neq \psi'$. □

# A Finite entropy and positive measures

The aim of this appendix is to clarify the representation of positive measures with finite entropy, a task that is nontrivial due to their limited integrability properties.

**Lemma 39.** *Let $q_T$ be an $\mathcal{F}_T$-measurable random variable with values in $\mathbb{R}_+$. Under the conditions $\mathbb{E}[h(q_T)] < +\infty$, $\mathbb{P}(\{q_T > 0\}) = 1$ and $\mathbb{E}[q_T] = 1$, there exists a unique progressively measurable process $Z^\star$ (with values in $\mathbb{R}^d$) such that*

$$q_T = 1 + \int_0^T q_s Z_s^\star \cdot \mathrm{d}W_s. \tag{214}$$

*It satisfies $\frac{1}{2}\mathbb{E}[\int_0^T q_s |Z_s^\star|^2 \mathrm{d}s] = \mathbb{E}[h(q_T)]$. Moreover, $q_T$ can be represented as*

$$q_T = \mathcal{E}_T\left(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s\right). \tag{215}$$

*Lastly, the process $(q_t := (\mathcal{E}_t(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s))_{t\in[0,T]}$ is true a martingale. It satisfies $\mathbb{E}[q_T^\star] < +\infty$, and is the unique solution to the SDE*

$$q_t = 1 + \int_0^t q_s Z_s^\star \cdot \mathrm{d}W_s, \quad t \in [0,T], \tag{216}$$

*within the class of continuous positive-valued and $\mathbb{F}$-adapted processes.*

As a consequence of the above lemma, the class $\mathcal{Q}$ is parameterized by the sole given data $q$ and $Y^\star$. Indeed, it suffices to apply the lemma above to the process $(\exp(-\int_0^t Y_s^\star \mathrm{d}s) q_t)_{t\in[0,T]}$ to obtain the representation in equation (4), namely

$$q_T = 1 + \int_0^T q_s Y_s^\star \mathrm{d}s + \int_0^T q_s Z_s^\star \cdot \mathrm{d}W_s.$$

*Proof. Step 1: representation of $q_T$ in the form (214).* Let $q_t = \mathbb{E}[q_T|\mathcal{F}_t]$, for $t \in [0,T]$. It is a strictly positive (continuous) martingale. By $L\log L$-Doob's maximal inequality, we deduce that $\mathbb{E}[q_T^*] < +\infty$.

For any integer $m \geq 1$, we let $q_T^m := q_T \wedge m$. By martingale representation theorem, we can write

$$q_T^m = \mathbb{E}[q_T^m] + \int_0^T Z_s^m \cdot \mathrm{d}W_s.$$

Letting $q_t^m = \mathbb{E}[q_T^m|\mathcal{F}_t]$, for $t \in [0,T]$, we invoke $L\log L$-Doob's maximal inequality again to deduce that there exists a universal constant $C > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |q_t^m - q_t|\right] \leq C\mathbb{E}\left[\max\left\{0, |q_T^m - q_T| \ln\left(|q_T^m - q_T|\right)\right\}\right].$$

Obviously (since $q_T^m = q_T$ if $q_T \leq m$),

$$|q_T^m - q_T| \ln\left(|q_T^m - q_T|\right) \leq q_T |\ln(2q_T)|.$$

By dominated convergence theorem, we deduce that

$$0 \leq \limsup_{m\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} |q_t^m - q_t|\right] \leq C \lim_{m\to\infty} \mathbb{E}\left[\max\left\{0, |q_T^m - q_T| \ln\left(|q_T^m - q_T|\right)\right\}\right] = 0.$$

In particular,

$$\lim_{m\to\infty}\sup_{k\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}|q_t^{m+k}-q_t^m|\right]=0.$$

By Bürkolder-Davies-Gundy inequality, we further obtain that

$$\lim_{m\to\infty}\sup_{k\in\mathbb{N}}\mathbb{E}\left[\left(\int_0^T|Z_t^m-Z_t^{m+k}|^2\mathrm{d}t\right)^{1/2}\right]=0.$$

By completeness of $L^1(\Omega\times[0,T],\mathcal{R},\mathbb{P}\otimes\mathrm{Leb}_{[0,T]})$ (where $\mathcal{R}$ is the progressive $\sigma$-field), we deduce that there exists a process $Z$ satisfying

$$\mathbb{E}\left[\left(\int_0^T|Z_t|^2\mathrm{d}t\right)^{1/2}\right]<+\infty,$$

and

$$q_T=1+\int_0^T Z_s\cdot\mathrm{d}W_s.$$

Since $(q_t)_{t\in[0,T]}$ is continuous and strictly positive, we can let $Z_t^\star=Z_t/q_t$. We then notice, from Itô's formula, that the process

$$\left(h(q_t)-\frac{1}{2}\int_0^t q_s|Z_s^\star|^2\mathrm{d}s\right)_{t\in[0,T]}$$

is a local martingale. In particular, one can find a sequence of stopping times $(\sigma_m)_{m\geq1}$, converging to $T$ as $m$ tends to $+\infty$, such that for all $m\geq1$,

$$\mathbb{E}\left[h\left(q_{T\wedge\sigma_m}\right)\right]=\frac{1}{2}\mathbb{E}\left[\int_0^{T\wedge\sigma_m}q_s|Z_s^\star|^2\mathrm{d}s\right]. \tag{217}$$

We now prove that

$$\sup_{m\geq1}\mathbb{E}\left[h\left(q_{T\wedge\sigma_m}\right)\right]<+\infty.$$

Since

$$q_{T\wedge\sigma_m}=\mathbb{E}\left[q_T|\mathcal{F}_{T\wedge\sigma_m}\right].$$

and because $h$ is convex, we deduce that

$$\mathbb{E}\left[h\left(q_{T\wedge\sigma_m}\right)\right]\leq\mathbb{E}[h(q_T)],$$

from which we get, by applying Fatou's lemma to the right-hand side on (217), that

$$\frac{1}{2}\mathbb{E}\left[\int_0^T q_s|Z_s^\star|^2\mathrm{d}s\right]\leq\mathbb{E}[h(q_T)].$$

Conversely, by observing that the function $h$ is lower bounded and then by applying Fatou's lemma to the left-hand side on (217), we get

$$\mathbb{E}[h(q_T)]\leq\frac{1}{2}\mathbb{E}\left[\int_0^T q_s|Z_s^\star|^2\mathrm{d}s\right].$$

By the last two identities, we deduce that the two terms on the above inequality are equal.

*Step 2: representation of $q_T$ in the form* (215). Recalling that $(q_t)_{t\in[0,T]}$, as defined in the first step, is continuous and strictly positive, we deduce that, a.s., it is lower bounded by a positive constant. This proves that, a.s.,

$$\int_0^T |Z_s^\star|^2 \mathrm{d}s < +\infty,$$

which makes it possible to define $(\mathcal{E}_t(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s))_{t\in[0,T]}$.

Moreover, one can compute the logarithm of the process $q$. By Itô's formula, we get

$$\ln(q_T) = -\frac{1}{2}\int_0^T |Z_s^\star|^2 \mathrm{d}s + \int_0^T Z_s^\star \cdot \mathrm{d}W_s,$$

which says that

$$q_T = \mathcal{E}_T\left(\int_0^\cdot Z_s^\star \cdot \mathrm{d}W_s\right).$$

This gives the expected representation of $q_T$, and more generally of the process $(q_t = \mathbb{E}[q_T|\mathcal{F}_t])_{t\in[0,T]}$.

Uniqueness to (216) can be easily checked by computing $(q_t'/q_t)_{t\in[0,T]}$ and then by proving that this ratio is constant (equal to 1), for any other solution $q' = (q_t')_{t\in[0,T]}$. □

# B A priori estimates on a linear SDE

Let $q \in \mathcal{Q}$ and $\psi \in \mathcal{A}$. In this section we provide technical results for controlled linear SDEs, including $S^1(\mathbb{F},\mathbb{Q})$ and $S^2(\mathbb{F},\mathbb{Q})$ regularity of the solutions, with $\mathbb{Q} = q\mathbb{P}$. With the same notations as in A1 and A2 , we consider the same generic controlled equation as in (7), namely

$$\mathrm{d}X_t = (a_t + b_t X_t + c_t\psi_t)\,\mathrm{d}t + (\nu_t + r\sigma_t(\psi_t))\,\mathrm{d}W_t, \quad X_0 = \eta. \tag{218}$$

Following the first step in the proof of Lemma 35, we denote by $\Gamma = (\Gamma_t)_{t\in[0,T]}$ the resolvent associated with the linear part of the equation, namely the solution (with values in the space of $n \times n$ matrices) of

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_t = b_t\Gamma_t, \quad t \in [0,T], \quad \Gamma_0 = I_n,$$

with $I_n$ standing for the $n \times n$ identity matrix. It is well-known that $\Gamma_t$ is invertible for any $t \in [0,T]$ and that the solution to (218) can be represented as

$$X_t = \Gamma_t\left(\eta + \int_0^t \Gamma_s^{-1}(a_s + c_s\psi_s)\,\mathrm{d}s + \int_0^t \Gamma_s^{-1}(\nu_s + r\sigma_s(\psi_s))\,\mathrm{d}W_s\right), \quad t \in [0,T].$$

Recalling assumptions A1 and A2, that is to say $\|\eta\|_{L^\infty(\mathcal{F}_0,\mathbb{R}^n)} + \|a\|_{L^\infty(\mathbb{F},\mathbb{R}^n)} + \|b\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})} + \|c\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})} + \|\nu\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})} + \|\sigma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})} < +\infty$, we deduce that there exists a constant $C > 0$, independent of $\psi$, such that

$$X_T^* \le C\left(1 + \int_0^T |\psi_s|\,\mathrm{d}s\right) + M_T^*, \quad M_t := \Gamma_t\int_0^t \Gamma_s^{-1}(\nu_s + r\sigma_s(\psi_s))\mathrm{d}W_s. \tag{219}$$

**Lemma 40.** *The unique solution $X$ to (218) belongs to $S^{2-r}(\mathbb{F},\mathbb{Q},\mathbb{R}^n)$, where $r\in\{0,1\}$ is as in A2, and there exists a constant $C$, independent of $q$ and $\psi$, such that*

$$\mathbb{E}\left[q_T|X_T^*|^{2-r}\right]\le C\left(1+\mathcal{S}(q)+\mathcal{S}^\star(\psi)\right).$$

*Proof.* We distinguish between the two cases $r=0$ and $r=1$.
*Step 1: case $r=0$.* We have that

$$|X_T^*|^2\le 2C\left(1+\int_0^T|\psi_s|^2\mathrm{d}s\right)+2\sup_{t\in[0,T]}\left|\Gamma_t\int_0^t\Gamma_s^{-1}\nu_s\mathrm{d}W_s\right|^2.$$

Using that $\|\nu\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})}+\|\Gamma^{-1}\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}<+\infty$ and [91, Theorem IV.37.8], we know that $M_T^*$ (whose square appears on the right-hand side) has sub-Gaussian tails, i.e., $M_T^*$ belongs to $L^{2,\vartheta}_{\exp}(\mathcal{F}_T,\mathbb{R})$ for a certain $\vartheta>0$. Therefore, by the duality inequalities (13) and (14), there exists a constant $C>0$ which might increase from line to line such that

$$\begin{aligned}\mathbb{E}\left[q_T|X_T^*|^2\right]&\le C\left(1+\mathbb{E}\left[q_T\int_0^T|\psi_s|^2\mathrm{d}s\right]+\mathbb{E}[h(q_T)]+\mathbb{E}\left[\exp(\vartheta(M_T^*)^2\right]\right)\\&\le C\left(1+\mathcal{S}(q)+\mathcal{S}^\star(\psi)\right)<+\infty.\end{aligned}$$

*Step 2: case $r=1$.* In this situation we have

$$|X_T^*|\le C\left(1+\int_0^T|\psi_s|\mathrm{d}s\right)+|M_T^*|.$$

Introducing $\tilde W_t=W_t-\int_0^t Z_s^\star\mathrm{d}s$ and applying Girsanov's theorem (see Lemma 39), we have

$$\begin{aligned}\mathbb{E}\left[q_T|X_T^*|\right]\le&C\mathbb{E}\left[q_T\left(1+\int_0^T|\psi_s|\mathrm{d}s+|M_T^*|\right)\right]\\\le&C\mathbb{E}\left[q_T\left(1+\int_0^T|\psi_s|\mathrm{d}s+\left|\sup_{t\in[0,T]}\Gamma_t\int_0^t\Gamma_s^{-1}(\nu_s+\sigma_s(\psi_s))\mathrm{d}\tilde W_s\right|\right)\right]\\&+C\mathbb{E}\left[q_T\left|\sup_{t\in[0,T]}\Gamma_t\int_0^t\Gamma_s^{-1}(\nu_s+\sigma_s(\psi_s))Z_s^\star\mathrm{d}s\right|\right].\end{aligned}\tag{220}$$

On the one hand, by Itô's isometry (under the measure $\mathbb{Q}:=\mathcal{E}_T(\int_0^\cdot Z_s^\star\cdot\mathrm{d}W_s)\mathbb{P}$) and the assumption A2 on $\sigma$, we have that

$$\mathbb{E}\left[q_T\left|\sup_{t\in[0,T]}\Gamma_t\int_0^t\Gamma_s^{-1}\sigma_s(\psi_s)\mathrm{d}\tilde W_s\right|\right]\le C\mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T\left[1+|\psi_s|^2\right]\mathrm{d}s\right)^{1/2}\right].\tag{221}$$

On the other hand, by Fenchel-Young inequality, we also have that

$$\mathbb{E}\left[q_T\left|\sup_{t\in[0,T]}\Gamma_t\int_0^t\Gamma_s^{-1}\sigma_s(\psi_s)Z_s^\star\mathrm{d}s\right|\right]\le C\mathbb{E}\left[q_T\int_0^T\left[1+|\psi_s|^2+|Z_s^\star|^2\right]\mathrm{d}s\right].\tag{222}$$

Combining (220) with (221) and (222), we obtain that

$$\begin{aligned}\mathbb{E}\left[q_T|X_T^*|\right] &\le C\mathbb{E}\left[q_T\left(1+\int_0^T|\psi_s|^2\mathrm{d}s+\int_0^T|Z_s^\star|^2\mathrm{d}s\right)\right]\\ &\le C\left(1+\mathbb{E}\left[q_T\int_0^T|\psi_s|^2\mathrm{d}s\right]+\mathbb{E}\left[h(q_T)\right]\right)\\ &\le C\left(1+\mathcal{S}(q)+\mathcal{S}^\star(\psi)\right)<+\infty,\end{aligned}$$

where the last two lines follow by duality inequality (14) between $\mathcal{S}$ and $\mathcal{S}^\star$, concluding the proof. □

In fact, the proof of Lemma 40 can be easily re-examined to get the following variant:

**Lemma 41.** *Let $r\in\{0,1\}$ be as in A2 and $X$ be the unique solution to (218). Then, for any $\varepsilon\in(0,1)$, there exist two constants $C_\varepsilon>0$ and $c_\varepsilon>0$, independent of $q$ and $\psi$, such that*

$$\mathbb{E}\left[q_T\,|X_T^*|\right]\le C_\varepsilon+c_\varepsilon\mathcal{S}(q)+\varepsilon\mathbb{E}\left[q_T\int_0^T|\psi_s|^2\mathrm{d}s\right],$$

*where the second constant is explicitly given by*

$$c_\varepsilon=2\beta e^{\alpha T}\|\Gamma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\Gamma^{-1}\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\left(\|\nu\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})}+\frac{3}{\varepsilon}e^{\alpha T}\|\sigma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d\times n})}\right).$$

*Proof.* It suffices to adapt the computations developed in the second step of the proof of Lemma 40 (whether $r$ is equal to 0 or 1). Throughout the proof, the value of $\varepsilon\in(0,1)$ is fixed. In the following, we shall use repeatedly that, for any $s\in[0,T]$, $\mathbb{E}[q_T]\le\exp(\alpha T)\mathbb{E}[q_s]$ and

$$\mathbb{E}\left[q_T\int_0^T|\psi_s|^2\mathrm{d}s\right]=\mathbb{E}\left[\int_0^T\mathbb{E}[q_T|\mathcal{F}_s]|\psi_s|^2\mathrm{d}s\right]\le e^{\alpha T}\mathbb{E}\left[\int_0^T q_s|\psi_s|^2\mathrm{d}s\right].$$

The first term on the right-hand side of (220) can be easily bounded, by means of Young's inequality. We obtain

$$\begin{aligned}\mathbb{E}\left[q_T\left(1+\int_0^T|\psi_s|\mathrm{d}s\right)\right]&\le\mathbb{E}\left[q_T\left(1+C_\varepsilon+\frac{\varepsilon}{3}\int_0^T|\psi_s|^2\mathrm{d}s\right)\right]\\ &\le C_\varepsilon+\frac{\varepsilon}{3}\mathbb{E}\left[\int_0^T q_s|\psi_s|^2\mathrm{d}s\right].\end{aligned}$$

Similarly, by Jensen's and Young's inequalities, (221) yields

$$\begin{aligned}&\mathbb{E}\left[q_T\left|\sup_{t\in[0,T]}\Gamma_t\int_0^t\Gamma_s^{-1}\sigma_s(\psi_s)\mathrm{d}\tilde{W}_s\right|\right]\\ \le&\mathbb{E}\left[q_T\left|\int_0^T\Gamma_t\Gamma_s^{-1}\sigma_s(\psi_s)\mathrm{d}s\right|^2\right]^{1/2}\\ \le&T\|\Gamma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\Gamma^{-1}\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\nu\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})}\mathbb{E}\left[q_T\right]^{1/2}\\ &+r\|\Gamma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\Gamma^{-1}\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\sigma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d\times n})}\mathbb{E}\left[q_T\int_0^T|\psi_s|^2\mathrm{d}s\right]^{1/2}\\ \le&C_\varepsilon+\frac{\varepsilon}{3}\mathbb{E}\left[\int_0^T q_s|\psi_s|^2\mathrm{d}s\right].\end{aligned}$$

And, (222) can be rewritten as

$$
\begin{aligned}
&\mathbb{E}\left[q_T\left|\sup_{t\in[0,T]}\Gamma_t\int_0^t\Gamma_s^{-1}\sigma_s(\psi_s)Z_s^\star \mathrm{d}s\right|\right]\\
\le&\|\Gamma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\Gamma^{-1}\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\nu\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})}\mathbb{E}\left[q_T\int_0^T|Z_s^\star|\mathrm{d}s\right]\\
&+\|\Gamma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\Gamma^{-1}\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\sigma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d\times n})}\mathbb{E}\left[q_T\int_0^T|\psi_s||Z_s^\star|\mathrm{d}s\right]\\
\le&e^{\alpha T}\|\Gamma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\Gamma^{-1}\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\nu\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})}\mathbb{E}\left[\int_0^T q_s(1+|Z_s^\star|^2)\mathrm{d}s\right]\\
&+\frac{3}{\varepsilon}e^{2\alpha T}\|\Gamma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\Gamma^{-1}\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\sigma\|_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d\times n})}\mathbb{E}\left[\int_0^T q_s|Z_s^\star|^2\mathrm{d}s\right]\\
&+\frac{\varepsilon}{3}\mathbb{E}\left[\int_0^T q_s|\psi_s|^2\mathrm{d}s\right]\\
\le&C+c_\varepsilon\mathcal{S}(q)+\frac{\varepsilon}{3}\mathbb{E}\left[\int_0^T q_s|\psi_s|^2\mathrm{d}s\right],
\end{aligned}
$$

for some fixed constant $C>0$. Combining (219) and the three last displays, we conclude that the desired inequality holds. □

**Lemma 42.** *Let $\psi=0$ and $X\in S^2(\mathbb{F},\mathbb{R}^n)$ be the associated solution to the state equation (218). Then $X_T^*\in S_{\exp}^{2-r,L\vartheta}(\mathcal{F}_T,\mathbb{R}^n)$ when $r=1$ in A2. The result holds for $r=0$ under the condition $4\vartheta L\|\Gamma\|^2_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\Gamma^{-1}\|^2_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\nu\|^2_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})}T<1$.*

*Proof.* Inequality (219) (with $\psi=0$) gives

$$X_T^*\le C+M_T^*,\quad \text{where}\quad M_t=\Gamma_t\int_0^t\Gamma_s^{-1}\nu_s\mathrm{d}W_s.$$

Raising to the power $2-r$ and multiplying by $\vartheta L$ both sides, then taking the exponential and the expectation both sides yields

$$\mathbb{E}\left[\exp(\vartheta L|X_T^*|^{2-r})\right]\le C\mathbb{E}\left[\exp\left(2\vartheta L|M_T^*|^{2-r}\right)\right],$$

for a constant $C>0$ which might have increased. We recall from [91, Theorem IV.37.8] that, for

$$\kappa:=\|\Gamma\|^2_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\Gamma^{-1}\|^2_{L^\infty(\mathbb{F},\mathbb{R}^{n\times n})}\|\nu\|^2_{L^\infty(\mathbb{F},\mathbb{R}^{n\times d})}T,$$

it holds $\mathbb{P}(\{M_T^\star\ge y\})\le\exp(-y^2/2\kappa)$ for all $y>0$, from which we deduce that $\mathbb{E}[\exp(2\vartheta L(M_T^*)^{2-r})]<+\infty$ when $r=1$. When $r=0$,

$$\mathbb{E}\left[\exp\left(2\vartheta L|M_T^*|^{2-r}\right)\right]\le 1+2\vartheta L\int_{\mathbb{R}}e^{y^2\left(2\vartheta L-\frac{1}{2\kappa}\right)}\mathrm{d}y,$$

which is finite whenever $\vartheta L\kappa<1/4$, hence concluding the proof. □

## C Uniform integrability and convergence results

This section focuses on technical results regarding the weak convergence of random variables, specifically addressing (weak) convergence in the $L\log L(\mathbb{F})$ space against random variables having a finite exponential moment, and vice versa. As we shall see, uniform integrability is key to establish convergence.

**Lemma 43.** *Let $(q^k)_{k\in\mathbb{N}}$ be a $\mathcal{Q}$-valued sequence, uniformly bounded in $L\log L(\mathbb{F})$, $(y^k)_{k\in\mathbb{N}}$ be an $L^\infty(\mathbb{F})$-valued sequence, uniformly bounded in $L^\infty(\mathbb{F})$, and $(\xi^k,\ell^k)_{k\in\mathbb{N}}$ be a collection of random variables in $L^1(\mathcal{F}_T)\times L^1(\mathbb{F})$ satisfying for all $\vartheta>0$,*

$$\sup_{k\in\mathbb{N}}\mathbb{E}\left[\exp\left(\vartheta|\xi_T^k|+\int_0^T\exp\left(\vartheta|\ell_s^k|\right)\mathrm{d}s\right)\right]<+\infty. \tag{223}$$

*Then, the sequences of random variables*

$$\left(q_T^k|y_T^k||\xi^k|\right)_{k\in\mathbb{N}},\quad\left(q_s^k|y_s^k||\ell_s^k|\right)_{k\in\mathbb{N}},$$

*are respectively uniformly integrable on $\Omega$ and $\Omega\times[0,T]$, equipped with $\mathbb{P}$ and $\mathbb{P}\otimes\mathrm{Leb}_{[0,T]}$, in the sense that*

$$\sup_{k\in\mathbb{N}}\mathbb{E}\left[q_T^k|y_T^k||\xi^k|\right]<+\infty,\quad\sup_{k\in\mathbb{N}}\mathbb{E}\left[\int_0^T q_s^k|y_s^k||\ell_s^k|\mathrm{d}s\right]<+\infty, \tag{224}$$

*and for every $\epsilon>0$, there exists $\delta>0$ such that*

$$\begin{aligned}\forall A\in\mathcal{F}_T,\quad\mathbb{P}(A)\le\delta\Rightarrow\mathbb{E}\left[q_T^k|y_T^k||\xi^k|\mathbb{1}_A\right]\le\varepsilon,\\ \forall A\in\mathcal{B}([0,T])\otimes\mathcal{F}_T,\ \left(\mathbb{P}\otimes\mathrm{Leb}_{[0,T]}\right)(A)\le\delta\Rightarrow\mathbb{E}\left[\int_0^T q_s^k|y_s^k||\ell_s^k|\mathbb{1}_A\mathrm{d}s\right]\le\varepsilon.\end{aligned} \tag{225}$$

*Proof.* Obviously, we can assume without any loss of generality that the processes $(y^k)_{k\in\mathbb{N}}$ are all equal to 1. Moreover, we just make the proof for the sequence $\left(q_s^k|y_s^k||\ell_s^k|\right)_{k\in\mathbb{N}}$, as the proof for $\left(q_T^k|y_T^n||\xi^k|\right)_{k\in\mathbb{N}}$ is analogous.

*Step 1: display (224) holds.* The proof of (224) follows from the duality inequality (13). If $\vartheta\ge1$ in the latter display, the term $\ln(\vartheta)x$ therein is positive, which leaves us with

$$x^\star x\le\frac{1}{\vartheta}h(x)+\exp(\vartheta x^\star), \tag{226}$$

for every $\vartheta\ge1$ and for all $x,x^\star>0$. We now apply this inequality with $x=q_T^k(\omega)$ and $x^\star=\int_0^T|\ell_s^k(\omega)|\mathrm{d}s$ for any $\omega\in\Omega$. Choosing $\vartheta=1$, we get

$$\mathbb{E}\left[q_T^k\int_0^T|\ell_s^k|\mathrm{d}s\right]\le\mathbb{E}\left[h\left(q_T^k\right)\right]+\mathbb{E}\left[\exp\left(\int_0^T|\ell_s^k|\mathrm{d}s\right)\right].$$

By assumption (see (223)), the right-hand side is uniformly bounded with respect to $k\in\mathbb{N}$. In order to derive (224), it suffices to recall, from the definition of $\mathcal{Q}$ that there exists a constant $C>0$ such that, for any $k\in\mathbb{N}$, $q_s^k\le C\mathbb{E}[q_T^k|\mathcal{F}_s]$ (with probability 1 under $\mathbb{P}$).

*Step 2: display (225) holds.* Fix $\epsilon>0$, and then choose $\vartheta\ge1$ large enough so that

$$\frac{1}{\vartheta}\sup_{k\in\mathbb{N}}\mathbb{E}\left[h\left(q_T^k\right)\right]\le\frac{\epsilon}{2CT},$$

with $C$ as in the first step. By (226) (with $x = |q_T^k|$ and $x^\star = |\ell_t^k|$), we deduce that, for $A \in \mathcal{B}([0,T]) \otimes \mathcal{F}_T$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T q_s^k |\ell_s^k| \mathbb{1}_A \mathrm{d}s\right] \\
&= \int_0^T \mathbb{E}\left[q_s^k |\ell_s^k| \mathbb{1}_A(s,\cdot)\right] \mathrm{d}s \\
&\leq \frac{C}{\vartheta} \int_0^T \mathbb{E}\left[h(q_T^k)\mathbb{E}\left[\mathbb{1}_A(s,\cdot)|\mathcal{F}_s\right]\right] \mathrm{d}s + \int_0^T \mathbb{E}\left[\exp\left(\vartheta|\ell_s^k|\right)\mathbb{E}\left[\mathbb{1}_A(s,\cdot)|\mathcal{F}_s\right]\right] \mathrm{d}s \\
&\leq \frac{\epsilon}{2} + \int_0^T \mathbb{E}\left[\exp\left(\vartheta|\ell_s^k|\right)\mathbb{E}\left[\mathbb{1}_A(s,\cdot)|\mathcal{F}_s\right]\right] \mathrm{d}s.
\end{aligned}$$

By Cauchy-Schwarz' and then Jensen's inequalities,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T q_s^k |\ell_s^k| \mathbb{1}_A \mathrm{d}s\right] \\
&\leq \frac{\epsilon}{2} + \left[\left(\mathrm{Leb}_{[0,T]} \otimes \mathbb{P}\right)(A)\right]^{1/2} \mathbb{E}\left[\int_0^T \exp\left(2\vartheta|\ell_s^k|\right) \mathrm{d}s\right]^{1/2} \\
&\leq \frac{\epsilon}{2} + \sqrt{T}\left[\left(\mathrm{Leb}_{[0,T]} \otimes \mathbb{P}\right)(A)\right]^{1/2} \mathbb{E}\left[\exp\left(2\vartheta \int_0^T |\ell_s^k| \mathrm{d}s\right)\right]^{1/2},
\end{aligned}$$

and we conclude by invoking (223). □

**Lemma 44.** *Let $(q^k)_{k\in\mathbb{N}} \in \mathcal{Q}$ be a sequence, weakly converging to $q$ for the $\sigma(L^1, L^\infty)$ topology. For $(\xi, \ell) \in L^1(\mathcal{F}_T) \times L^1(\mathbb{F})$, assume further that the sequences $(q_T^k \xi)_{k\in\mathbb{N}}$ and $(q_s^k \ell_s)_{k\in\mathbb{N}}$ are uniformly integrable. Then,*

$$\lim_{k\to+\infty} \mathbb{E}\left[q_T^k \xi + \int_0^T q_s^k \ell_s \mathrm{d}s\right] = \mathbb{E}\left[q_T \xi + \int_0^T q_s \ell_s \mathrm{d}s\right]. \tag{227}$$

*Proof.* We only show the convergence of $(q_s^k \ell_s)_{k\in\mathbb{N}}$, the proof for $(q_T^k \xi)_{k\in\mathbb{N}}$ being analogous. Let $(\tilde{\xi}, \tilde{\ell}) \in L^\infty(\mathbb{F}) \times L^\infty(\mathbb{F})$. By weak convergence, we have

$$\lim_{k\to+\infty} \mathbb{E}\left[\int_0^T q_s^k \tilde{\ell}_s \mathrm{d}s\right] = \mathbb{E}\left[\int_0^T q_s \tilde{\ell}_s \mathrm{d}s\right]. \tag{228}$$

Now, by uniform integrability (see Lemma 43), we also have

$$\lim_{a\to+\infty} R^a = 0, \quad \text{with} \quad R^a := \sup_{k\in\mathbb{N}} \mathbb{E}\left[\int_0^T q_s^k \ell_s \mathbb{1}_{\{|\ell_s|\geq a\}} \mathrm{d}s\right].$$

With this notation, we have, for any $a > 0$,

$$\limsup_{k\to+\infty} \mathbb{E}\left[\int_0^T q_s^k \ell_s \mathrm{d}s\right] \leq \limsup_{k\to+\infty} \mathbb{E}\left[\int_0^T q_s^k \ell_s \mathbb{1}_{\{|\ell_s|\leq a\}} \mathrm{d}s\right] + R^a. \tag{229}$$

Here, we can use (228) in order to identify the superior limit in the right-hand side. And then, letting $a \to +\infty$, we deduce from (229):

$$\limsup_{k\to+\infty} \mathbb{E}\left[\int_0^T q_s^k \ell_s \mathrm{d}s\right] \leq \mathbb{E}\left[\int_0^T q_s \ell_s \mathrm{d}s\right]. \tag{230}$$

Changing $\ell$ into $-\ell$, we also have

$$\liminf_{k\to+\infty}\mathbb{E}\left[\int_0^T q_s^k\ell_s\mathrm{d}s\right]\geq\mathbb{E}\left[\int_0^T q_s\ell_s\mathrm{d}s\right]. \tag{231}$$

Combining (230) and (231) yields that

$$\lim_{k\to+\infty}\mathbb{E}\left[\int_0^T q_s^k\ell_s\mathrm{d}s\right]=\mathbb{E}\left[\int_0^T q_s\ell_s\mathrm{d}s\right],$$

which concludes the proof □

# D Distance and differentiability on spaces of non-negative measures

## D.1 Generalized Wasserstein distance

We here establish the equivalence between the notion of continuity used in Section 4.1 and the notion of generalized $p$-Wasserstein distance introduced in [88] (see also [43]). We recall the following definition (using the notations introduced in Section 4.1, in particular the distances $d_p$ and $W_p$):

**Definition 45.** *Let $p\geq 1$ and $\mu,\nu\in\mathcal{M}_p(\mathbb{R}^n)$. We call generalized $p$-Wasserstein distance between $\mu$ and $\nu$ the quantity*

$$W_{p,\mathrm{ext}}(\mu,\nu):=\inf_{\tilde\mu,\tilde\nu\in\mathcal{M}_p(\mathbb{R}^n),\,\tilde\mu(\mathbb{R}^n)=\tilde\nu(\mathbb{R}^n)}\left\{W_p(\tilde\mu,\tilde\nu)+\|\mu-\tilde\mu\|_{\mathrm{TV}}+\|\nu-\tilde\nu\|_{\mathrm{TV}}\right\}.$$

The fact that $W_{p,\mathrm{ext}}$ is a distance is established in [88, Proposition 1]. We state below the main result of this section. To do so, we recall that a subset $E\subset\mathcal{M}_p(\mathbb{R}^n)$ is said to be $p$-uniformly integrable if

$$\sup_{\mu\in E}\int_{\mathbb{R}^n}(1+|x|^p)\mathrm{d}\mu(x)<+\infty,\qquad\lim_{a\to+\infty}\sup_{\mu\in E}\int_{\mathbb{R}^n}\mathbf{1}_{\{|x|\geq a\}}|x|^p\mathrm{d}\mu(x)=0.$$

We claim

**Proposition 46.** *For a given $p\geq 1$, let $\psi:\mathcal{M}_p(\mathbb{R}^n)\to\mathbb{R}$. Then, $\psi$ is continuous with respect to $W_{p,\mathrm{ext}}$ on any subset of $p$-uniform integrability, if and only the following two properties hold:*

1. *$\psi$ is continuous with respect to $d_p$, uniformly on subsets of $p$-uniform integrability;*
2. *on any isomass subset of $\mathcal{M}_p(\mathbb{R}^n)$, $\psi$ is continuous with respect to $W_p$.*

Pay attention that continuity of $\psi$ is just restricted to subsets that are $p$-uniformly integrable: equivalently, we require that $\psi(\mu_m)\to\psi(\mu)$ for any sequence $(\mu_m)_{m\geq1}$ that converges to $\mu$ in $\mathcal{M}_p(\mathbb{R}^n)$ with respect to $W_{p,\mathrm{ext}}$ and that is $p$-uniformly integrable (recall that convergence in $\mathcal{M}_p(\mathbb{R}^n)$ does not guarantee $p$-uniform integrability).

*Proof.* We first prove the implication (direct sense). We thus assume that $\psi$ is continuous with respect to $W_{p,\text{ext}}$ on any subset of $p$-uniform integrability. Obviously, any sequence that converges with respect to $W_p$ on an isomass subset of $\mathcal{M}_p(\mathbb{R}^n)$ is $p$-uniformly integrable and converges with respect to $W_{p,\text{ext}}$. Therefore, $\psi$ is continuous with respect to $W_p$ on any isomass subset. This is item 2 in the statement. In order to prove item 1, consider a sequence $(\mu_m)_{m\geq 1}$ that converges to some limit $\mu$, in $\mathcal{M}_p(\mathbb{R}^n)$ equipped with $d_p$. Clearly, it is $p$-uniformly integrable. Moreover, by [88, Theorem 3], $(\mu_m)_{m\geq 1}$ converges to $\mu$ with respect to $W_{p,\text{ext}}$. By continuity of $\psi$ with respect to $W_{p,\text{ext}}$, this shows that $\psi(\mu_m) \to \psi(\mu)$ as $m \to +\infty$. Continuity is uniform on any subset of $p$-uniform integrability. This follows from a standard compactness argument, as any subset of $p$-uniform integrability is relatively compact for $W_{p,\text{ext}}$.

We now establish the converse, assuming that $\psi$ satisfies items 1 and 2 in the statement. We thus consider a $p$-uniformly integrable sequence $(\mu_m)_{m\geq 1}$ that converges to some limit $\mu$, in $\mathcal{M}_p(\mathbb{R}^n)$ equipped with $\mathcal{W}_{p,\text{ext}}$.

If $\mu(\mathbb{R}^n) = 0$, then [88, Theorem 4] says that $(\mu^m)_{m\geq 1}$ converges to the null measure in $\mathcal{M}_p(\mathbb{R}^n)$. By item 1 in the statement, we deduce that $\psi(\mu^m) \to \psi(\mu)$ as $m \to +\infty$, as expected.

We thus assume that $\mu(\mathbb{R}^n) > 0$. By definition of $W_{p,\text{ext}}$, we can find two sequences $(\tilde{\mu}^m)_{m\geq 1}$ and $(\tilde{\nu}^m)_{m\geq 1}$ such that, for each $m \geq 1$, $\tilde{\mu}^m$ is dominated by $\mu^m$, $\tilde{\nu}^m$ is dominated by $\mu$, and $\tilde{\mu}^m(\mathbb{R}^n) = \tilde{\nu}^m(\mathbb{R}^n)$, and

$$\lim_{m\to\infty} \left[W_p(\tilde{\mu}^m, \tilde{\nu}^m) + \|\mu^m - \tilde{\mu}^m\|_{\text{TV}} + \|\mu - \tilde{\nu}^m\|_{\text{TV}}\right] = 0.$$

In particular, $\mu^m(\mathbb{R}^n) \to \mu(\mathbb{R}^n)$. Therefore, without any loss of generality, we can assume that $\mu^m(\mathbb{R}^n) > 0$ for any $m \geq 1$, which makes it possible to let

$$\bar{\mu}^m(\cdot) = \frac{\mu(\mathbb{R}^n)}{\tilde{\mu}^m(\mathbb{R}^n)}\tilde{\mu}^m(\cdot), \quad \bar{\nu}^m(\cdot) = \frac{\mu(\mathbb{R}^n)}{\tilde{\mu}^m(\mathbb{R}^n)}\tilde{\nu}^m(\cdot).$$

It is easy to see that $W_p(\bar{\mu}^m, \bar{\nu}^m)$ tends to 0 as $m \to +\infty$. Moreover, the sequence $(\bar{\mu}^m)_{m\geq 1}$ is $p$-uniformly integrable because $(\mu^m)_{m\geq 1}$ is $p$-uniformly integrable, and each $\bar{\mu}^m$ is dominated by $C\mu^m$, for a constant $C$ independent of $m$. Obviously, $(\bar{\nu}^m)_{m\geq 1}$ is also $p$-uniformly integrable (because $\bar{\nu}^m$ is dominated by $C\mu$, for a possibly different value of $C$, but still independent of $m$). Since $\psi$ is $W_p$-continuous on the subset of measures with constant mass equal to $\mu(\mathbb{R}^n)$, it is in particular equicontinuous on any subset of $p$-uniformly integrable measures with constant mass equal to $\mu(\mathbb{R}^n)$. Therefore, by item 2,

$$\lim_{m\to+\infty} |\psi(\bar{\mu}^m) - \psi(\bar{\nu}^m)| = 0.$$

It remains to see that $\|\mu^m - \bar{\mu}^m\|_{\text{TV}} \to 0$ as $m \to +\infty$. Since the two sequences $(\mu^m)_{m\geq 1}$ and $(\bar{\mu}^m)_{m\geq 1}$ are $p$-uniformly integrable, we deduce that $d_p(\mu^m, \tilde{\mu}^m) \to 0$ as $m$ tends to $+\infty$. By item 1 (using the fact that continuous is uniform on subsets of uniform $p$-integrability), we deduce that

$$\lim_{m\to+\infty} |\psi(\mu^m) - \psi(\bar{\mu}^m)| = 0.$$

Similarly,

$$\lim_{m\to+\infty} |\psi(\mu) - \psi(\bar{\nu}^m)| = 0.$$

By combining the last three displays, we complete the proof. $\square$

## D.2 Differentiability

The purpose of this subsection is to prove Lemma 14.

*Proof of Lemma 14.* To simplify, we prove the result assuming that A9 holds true without any restriction on the mass of $\mu$.

We first establish (55). Given $\mu \in \mathcal{M}_{2-r}(\mathbb{R}^n)$, we deduce from (42) (together with the continuity of the derivative in $d_{2-r}$) that, for any $t > 0$ and any $x \in \mathbb{R}^n$,

$$G\left(\mu + t\delta_x\right) = t \int_0^1 \frac{\delta G}{\delta \mu}\left(\mu + \theta t \delta_x, x\right) \mathrm{d}\theta.$$

And then, for any integer $\ell \geq 1$, any $t_1, \dots, t_\ell > 0$ and any $x_1, \dots, x_\ell \in \mathbb{R}^n$,

$$\begin{aligned}
&G\left(\mu + \sum_{i=1}^{\ell} t_i \delta_{x_i}\right) - G(\mu) \\
&= \sum_{i=1}^{\ell} \left[ G\left(\mu + \sum_{j=1}^{i} t_j \delta_{x_j}\right) - G\left(\mu + \sum_{j=1}^{i-1} t_j \delta_{x_j}\right)\right] \\
&= \sum_{i=1}^{\ell} t_i \int_0^1 \frac{\delta G}{\delta \mu}\left(\mu + \sum_{j=1}^{i-1} t_j \delta_{x_j} + \theta t_i \delta_{x_i}, x_i\right) \mathrm{d}\theta \\
&= \int_0^1 \left[ \int_{\mathbb{R}^n} \frac{\delta G}{\delta \mu}\left(\mu + \sum_{j=1}^{i-1} t_j \delta_{x_j} + \theta t_i \delta_{x_i}, y\right) \mathrm{d}\left(\sum_{i=1}^{\ell} t_i \delta_{x_i}\right)(y)\right] \mathrm{d}\theta.
\end{aligned}$$

By continuity of $\delta G/\delta \mu$ in the measure argument, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}_{|\varepsilon=0+} G\left(\mu + \varepsilon \sum_{i=1}^{\ell} t_i \delta_{x_i}\right) = \int_{\mathbb{R}^n} \frac{\delta G}{\delta \mu}(\mu, y)\, \mathrm{d}\left(\sum_{i=1}^{\ell} t_i \delta_{x_i}\right)(y).$$

And then,

$$\begin{aligned}
&G\left(\mu + \sum_{i=1}^{\ell} t_i \delta_{x_i}\right) - G(\mu) \\
&= \int_0^1 \left[ \int_{\mathbb{R}^n} \frac{\delta G}{\delta \mu}\left(\mu + \theta \sum_{i=1}^{\ell} t_i \delta_{x_i}, y\right) \mathrm{d}\left(\sum_{i=1}^{\ell} t_i \delta_{x_i}\right)(y)\right] \mathrm{d}\theta.
\end{aligned}$$

Now, we can approximate any given $\nu \in \mathcal{M}_{2-r}(\mathbb{R}^n)$ by measures $(\nu^\ell)_{\ell \geq 1}$ with the same mass, but with each being supported by a finite set; the approximation holds true with respect to $W_{2-r}$. For each $\ell \geq 1$, we have

$$G(\mu + \nu^\ell) - G(\mu) = \int_0^1 \left[ \int_{\mathbb{R}^n} \frac{\delta G}{\delta \mu}\left(\mu + \theta \nu^\ell, y\right) \mathrm{d}\nu^\ell(y)\right] \mathrm{d}\theta. \tag{232}$$

We denote by $\pi^\ell$ an optimal coupling between $\nu$ and $\nu^\ell$. Using the third line in (44), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \frac{\delta G}{\delta \mu}\left(\mu+\theta\nu^\ell, y\right) \mathrm{d}\left(\nu^\ell-\nu\right)(y)\right| \\
&\leq \int_{\mathbb{R}^n\times\mathbb{R}^n} \left|\frac{\delta G}{\delta \mu}\left(\mu+\theta\nu^\ell, y\right) - \frac{\delta G}{\delta \mu}\left(\mu+\theta\nu^\ell, z\right)\right| \mathrm{d}\pi^\ell(y,z) \\
&\leq C\left(1+M_{2-r}(\mu+\nu^\ell)\right)\int_{\mathbb{R}^n\times\mathbb{R}^n} (1+|y|^{1-r}+|z|^{1-r})|y-z|\mathrm{d}\pi^\ell(y,z).
\end{aligned}$$

Observing that the moments $(M_{2-r}(\nu^\ell))_{\ell\geq 1}$ are uniformly bounded (because the convergence holds true with respect to $W_{2-r}$) and using Cauchy-Schwarz inequality to handle the last term in the right-hand side when $r=0$, we deduce that the left-hand side in the above display tends to 0 as $\ell$ tends to $+\infty$.

Using the continuity of $\delta G/\delta\mu$ in $\mu$ with respect to $W_{2-r}$ on isomass subsets, and the growth condition (44), we can pass to the limit in (232). We get

$$G(\mu+\nu)-G(\mu)=\int_0^1\left[\int_{\mathbb{R}^n}\frac{\delta G}{\delta\mu}(\mu+\theta\nu,y)\,\mathrm{d}\nu(y)\right]\mathrm{d}\theta.$$

And then, for any $\varepsilon\in[0,1]$,

$$\begin{aligned}
G\left((1-\varepsilon)\mu+\varepsilon\nu\right)-G\left((1-\varepsilon)\mu\right) &= \varepsilon\int_0^1\left[\int_{\mathbb{R}^n}\frac{\delta G}{\delta\mu}\left((1-\varepsilon)\mu+\varepsilon\theta\nu,y\right)\mathrm{d}\nu(y)\right]\mathrm{d}\theta \\
&= \varepsilon\int_0^1\left[\int_{\mathbb{R}^n}\frac{\delta G}{\delta\mu}(\mu,y)\,\mathrm{d}\nu(y)\right]\mathrm{d}\theta + o(\varepsilon),
\end{aligned}$$

where $o(\varepsilon)/\varepsilon\to 0$ as $\varepsilon$ tends to 0, with the last line following from (45). Performing a similar expansion for $\nu=\mu$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|_{\varepsilon=0+}G\left((1-\varepsilon)\mu+\varepsilon\nu\right)=\int_{\mathbb{R}^n}\frac{\delta G}{\delta\mu}(\mu,y)\,\mathrm{d}\left[\nu-\mu\right](y),$$

from which we deduce that

$$G(\nu)-G(\mu)=\int_0^1\left[\int_{\mathbb{R}^n}\frac{\delta G}{\delta\mu}\left((1-\theta)\mu+\theta\nu,y\right)\mathrm{d}\left[\nu-\mu\right](y)\right]\mathrm{d}\theta.$$

Choosing $\nu=(q'\mathbb{P}_X$ and $\mu=(q\mathbb{P}_X)$, this completes the proof of (55).

It remains to prove (56). Generally speaking, it is a consequence of [35, Proposition 5.44], applied on the space $(\Omega,\mathcal{F},q\mathbb{P}/\mathbb{E}(q))$. Indeed, following Remark 13, we can apply [35, Proposition 5.44] to the function $X\in L^2(\Omega,\mathcal{F},q\mathbb{P}/c)\mapsto G^{(c)}((q\mathbb{P}/c)_X)$, where $c:=\mathbb{E}[q]$ and $G^{(c)}(\mu)=G(c\mu)$. We obtain (56) when $X$ satisfies $\mathbb{E}[q|X|^2]<+\infty$. When $X$ is just in $L^1(\Omega,\mathcal{F},q\mathbb{P})$, we can approximate it, in $L^1(q\mathbb{P})$, by a sequence in $L^2(\Omega,\mathcal{F},q\mathbb{P})$; we then apply (56) to the approximating subsequence and then pass to the limit using (44) and (46). □

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